FORCE-type schemes for hyperbolic conservation laws.

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The big picture: numerical methods to solve

$$\partial_t Q + \partial_x F(Q) + \partial_y G(Q) + \partial_z H(Q) = S(Q) + D(Q)$$

$$\partial_t Q + A(Q)\partial_x Q + B(Q)\partial_y Q + C(Q)\partial_z Q = S(Q) + D(Q)$$

Source terms S(Q) may be stiff

Advective terms may not admit a conservative form (nonconservative products)

Meshes are assumed unstructured

Very high order of accuracy in both space and time

May use upwind or centred approaches for numerical fluxes

Recall the integral form of the conservation laws

$$\partial_t Q + \partial_x F(Q) = 0$$

in a control volume $[x_L, x_R] \times [t_1, t_2]$ is

$$\int_{x_L}^{x_R} Q(x,t_2)dx = \int_{x_L}^{x_R} Q(x,t_1)dx - \left[\int_{t_1}^{t_2} F(Q(x_R,t))dt - \int_{t_1}^{t_2} F(Q(x_L,t))dt\right]$$

$$\frac{1}{\Delta x}\int_{x_L}^{x_R} Q(x,t_2)dx = \frac{1}{\Delta x}\int_{x_L}^{x_R} Q(x,t_1)dx - \frac{1}{\Delta x}\Delta t\left[\frac{1}{\Delta t}\int_{t_1}^{t_2} F(Q(x_R,t))dt - \frac{1}{\Delta t}\int_{t_1}^{t_2} F(Q(x_L,t))dt\right]$$

$$\Rightarrow Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$

Conservative schemes in 1D

$$\partial_t Q + \partial_x F(Q) = 0$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$

Task: define numerical flux

$$F_{i+1/2}$$

Basic property required: MONOTONICITY

There are two approaches:

I: Upwind approach. Solve the Riemann problem

$$\left.\begin{array}{l} \partial_t Q + \partial_x F(Q) = 0 \\ Q(x,0) = \begin{cases} Q_i^n & \text{if} \quad x < 0 \\ Q_{i+1}^n & \text{if} \quad x > 0 \end{cases} \end{array}\right\} \Rightarrow F_{i+1/2}$$

II: Centred approach. The numerical flux is

$$F_{i+1/2} = H(Q_i^n, Q_{i+1}^n)$$

Properties required from 2-point flux

$$F_{i+1/2} = H(U,V)$$

Consistency: $F_{i+1/2} = H(U,U) = F(U)$

Monotonicity:

$$f_{i+1/2} = h(q_i^n, q_{i+1}^n) \rightarrow q_i^{n+1} = L(\ldots q_{i-1}^n, q_i^n, q_{i+1}^n \ldots)$$

Definition: a monotone scheme satisfies

$$\frac{\partial}{\partial q_k^n} L(\ldots q_{i-1}^n, q_i^n, q_{i+1}^n \ldots) \geq 0 \ \forall k$$

Properties required from 2-point flux

Remark: for a linear scheme $q_i^{n+1} = \sum_{k=-l}^{r} \beta_k q_{i+k}^n$

monotonicity requires positivity of coefficients: $\beta_k \geq 0 \ \forall k$

Theorem: for a two-point flux, necessary conditions for monotonicity are

$$\frac{\partial}{\partial u} f_{i+1/2} = \frac{\partial}{\partial u} h(u, v) \geq 0; \quad \frac{\partial}{\partial v} f_{i+1/2} = \frac{\partial}{\partial v} h(u, v) \leq 0$$

Classical centred numerical fluxes

The Lax-Friedrichs flux

$$F_{i+1/2}^{LF} = \frac{1}{2}\left(F(Q_i^n) + F(Q_{i+1}^n)\right) - \frac{1}{2}\frac{\Delta x}{\Delta t}\left(Q_{i+1}^n - Q_i^n\right)$$

Properties

1. Linearly stable for $0 \le |c| \le 1$
2. Monotone for all CFL numbers in the stability range
3. Largest local truncation error of all monotone schemes

$c = \Delta t \lambda / \Delta x$ the Courant number

Classical centred numerical fluxes, contin...

The Lax-Wendroff flux (2 versions)

$$F^{LW}_{i+1/2} = F(Q^{lw}_{i+1/2}), \quad Q^{lw}_{i+1/2} = \frac{1}{2}\left(Q^n_i + Q^n_{i+1}\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q^n_{i+1}) - F(Q^n_i)\right)$$

$$F^{LW}_{i+1/2} = \frac{1}{2}\left(F(Q^n_i) + (Q^n_{i+1})\right) - \frac{1}{2}\frac{\Delta t}{\Delta x} A_{i+1/2}\left(F(Q^n_{i+1}) - F(Q^n_i)\right)$$

Properties

1. Linearly stable for $0 \leq |c| \leq 1$
2. Non-monotone (oscillatory)
3. Second-order accurate in space and time

Classical centred numerical fluxes, contin...

The Godunov centred flux (1961)

$$F_{i+1/2}^{GC} = F(Q_{i+1/2}^{gc}), \quad Q_{i+1/2}^{gc} = \frac{1}{2}\left(Q_i^n + Q_{i+1}^n\right) - \frac{\Delta t}{\Delta x}\left(F(Q_{i+1}^n) - F(Q_i^n)\right)$$

Properties

1. Linearly stable for $0 \leq |c| \leq \frac{1}{2}\sqrt{2}$

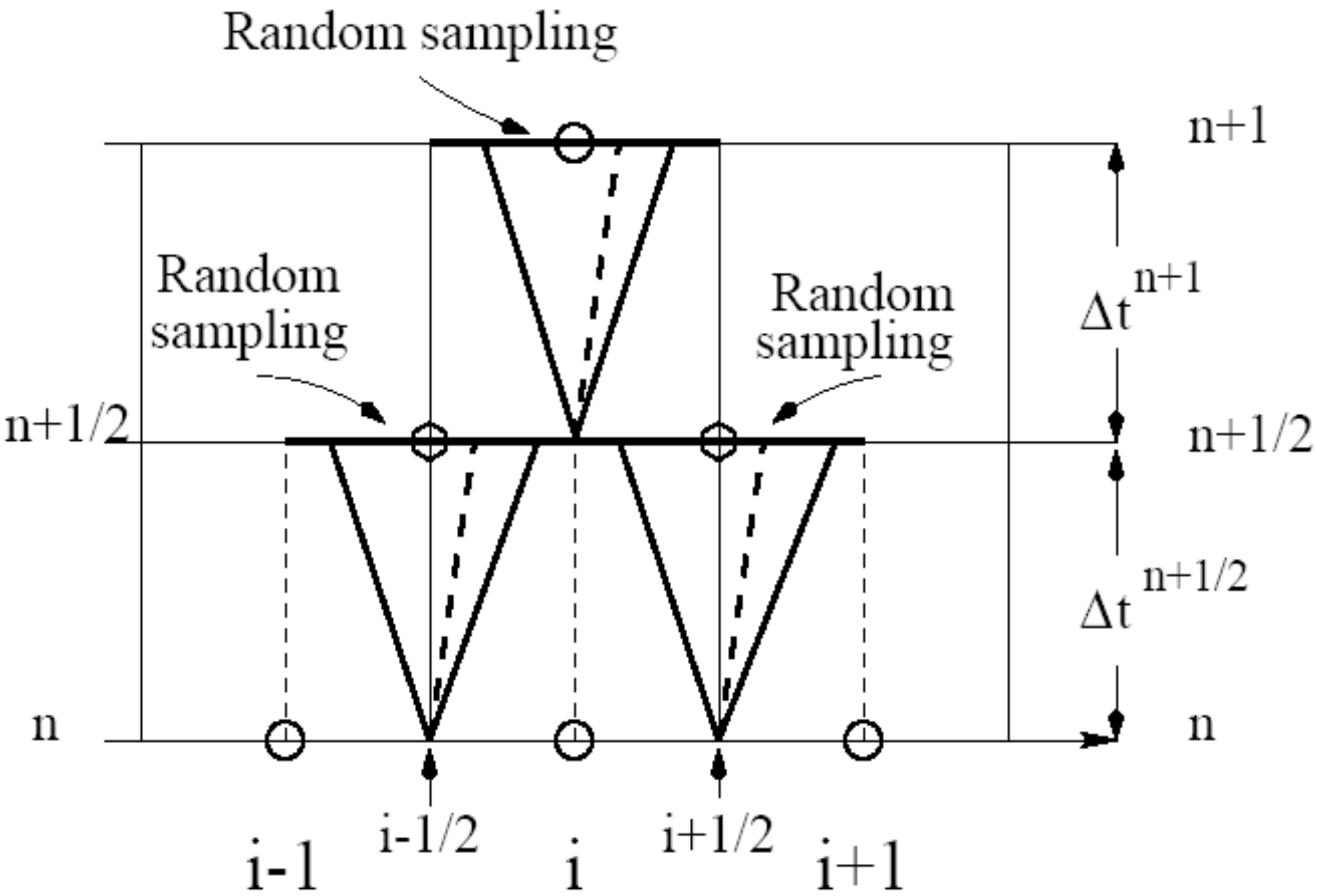
2. Monotone for $\frac{1}{2} \leq |c| \leq \frac{1}{2}\sqrt{2}$

3. Non-monotone for $0 \leq |c| \leq \frac{1}{2}$

The FORCE flux
(First ORder CEntred)

Toro E F.
On Glimm-related schemes for conservation laws.
Technical Report MMU-9602, Department of Mathematics and Physics, Manchester Metropolitan University, 1996,UK

Glimm's method on a staggered mesh



Recall the integral form of the conservation laws

$$\partial_t Q + \partial_x F(Q) = 0$$

in a control volume $\left[x_L, x_R\right] \times \left[t_1, t_2\right]$

$$\frac{1}{\Delta x}\int_{x_L}^{x_R} Q(x,t_2)dx = \frac{1}{\Delta x}\int_{x_L}^{x_R} Q(x,t_1)dx - \frac{1}{\Delta x}\Delta t\left[\frac{1}{\Delta t}\int_{t_1}^{t_2} F(Q(x_R,t))dt - \frac{1}{\Delta t}\int_{t_1}^{t_2} F(Q(x_L,t))dt\right]$$

Step I

$$Q_{i+1/2}^{n+1/2} = \frac{1}{2}\left(Q_i^n + Q_{i+1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q_{i+1}^n) - F(Q_i^n)\right)$$

$$Q_{i-1/2}^{n+1/2} = \frac{1}{2}\left(Q_{i-1}^n + Q_i^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q_i^n) - F(Q_{i-1}^n)\right)$$

Step II

$$Q_i^{n+1} = \frac{1}{2}\left(Q_{i-1/2}^{n+1/2} + Q_{i+1/2}^{n+1/2}\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q_{i+1/2}^{n+1/2}) - F(Q_{i-1/2}^{n+1/2})\right)$$

Question: can we write

$$Q_i^{n+1} = \frac{1}{2}\left(Q_{i-1/2}^{n+1/2} + Q_{i+1/2}^{n+1/2}\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q_{i+1/2}^{n+1/2}) - F(Q_{i-1/2}^{n+1/2})\right)$$

as a one-step conservative method

$$\mathrm{Q}_{\mathrm{i}}^{\mathrm{n+1}} = \mathrm{Q}_{\mathrm{i}}^{\mathrm{n}} - \frac{\Delta \mathrm{t}}{\Delta \mathrm{x}}\left(\mathrm{F}_{\mathrm{i+1/2}}^{\mathrm{force}} - \mathrm{F}_{\mathrm{i-1/2}}^{\mathrm{force}}\right)$$

with a given numerical flux

$$\mathrm{F}_{\mathrm{i+1/2}}^{\mathrm{force}}$$

Answer: YES

The numerical flux is

$$F^{force}_{i+1/2} = \frac{1}{4}\left[F^n_i + 2F(Q^{n+1/2}_{i+1/2}) + F^n_{i+1} - \frac{\Delta x}{\Delta t}(Q^n_{i+1} - Q^n_i)\right]$$

But recall

$$F^{LW}_{i+1/2} = F(Q^{lw}_{i+1/2}), \quad Q^{lw}_{i+1/2} = \frac{1}{2}\left(Q^n_i + Q^n_{i+1}\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q^n_{i+1}) - F(Q^n_i)\right)$$

$$F^{LF}_{i+1/2} = \frac{1}{2}\left(F(Q^n_i) + F(Q^n_{i+1})\right) - \frac{1}{2}\frac{\Delta x}{\Delta t}\left(Q^n_{i+1} - Q^n_i\right)$$

The numerical flux is in fact

$$F_{i+1/2}^{force} = \frac{1}{2}\left(F_{i+1/2}^{LW} + F_{i+1/2}^{LF}\right)$$

with

$$F_{i+1/2}^{LW} = F(Q_{i+1/2}^{lw}), \quad Q_{i+1/2}^{lw} = \frac{1}{2}\left(Q_i^n + Q_{i+1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q_{i+1}^n) - F(Q_i^n)\right)$$

$$F_{i+1/2}^{LF} = \frac{1}{2}\left(F(Q_i^n) + F(Q_{i+1}^n)\right) - \frac{1}{2}\frac{\Delta x}{\Delta t}\left(Q_{i+1}^n - Q_i^n\right)$$

Properties of the FORCE scheme

$$\partial_t q + \lambda \partial_x q = 0$$

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}\left[f_{i+1/2} - f_{i-1/2}\right]$$

$$f_{i+1/2}^{force} = \frac{(1+c)^2}{4c}(\lambda q_i^n) + \frac{(1-c)^2}{4c}(\lambda q_{i+1}^n)$$

$$q_i^{n+1} = b_{-1} q_{i-1}^n + b_0 q_i^n + b_1 q_{i+1}^n$$

$$b_{-1} = \frac{1}{4}(1+c)^2 \quad b_0 = \frac{1}{2}(1-c^2) \quad b_{-1} = \frac{1}{4}(1-c)^2$$

Properties of the FORCE scheme, cont.

$$Stable \quad 0 \leq |c| \leq 1$$

$$Monotone$$

$$Modified\ equation\ \partial_t q + \lambda \partial_x q = \alpha_{force} \partial_x^{(2)} q$$

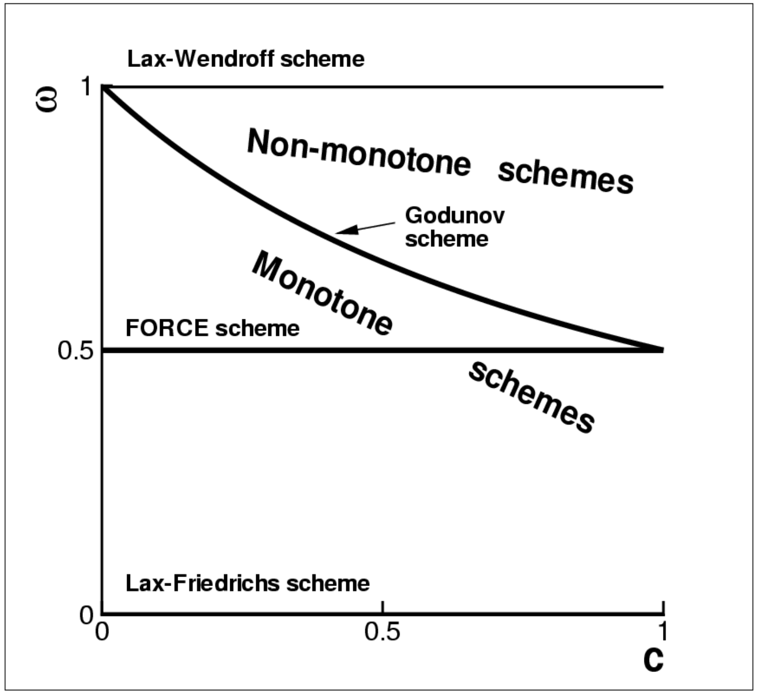
$$\alpha_{force} = \frac{1}{4} \lambda \Delta x \left(\frac{1 - c^2}{c} \right) = \frac{1}{2} \alpha_{lf}$$

Proof of convergence of FORCE scheme in:

Chen C Q and Toro E F.
Centred schemes for non-linear hyperbolic equations.
J Hyperbolic . Differential. Equations. 1 (1), pp 531-566, 2004.

The FORCE flux for the scalar case: more general averaging.

$$F^{\omega}_{i+1/2} = \omega F^{LW}_{i+1/2} + (1-\omega) F^{LF}_{i+1/2}\,, \quad 0 < \omega < 1$$



Special cases:

$$\omega = 0 \text{ (Lax - Friedrichs)}$$

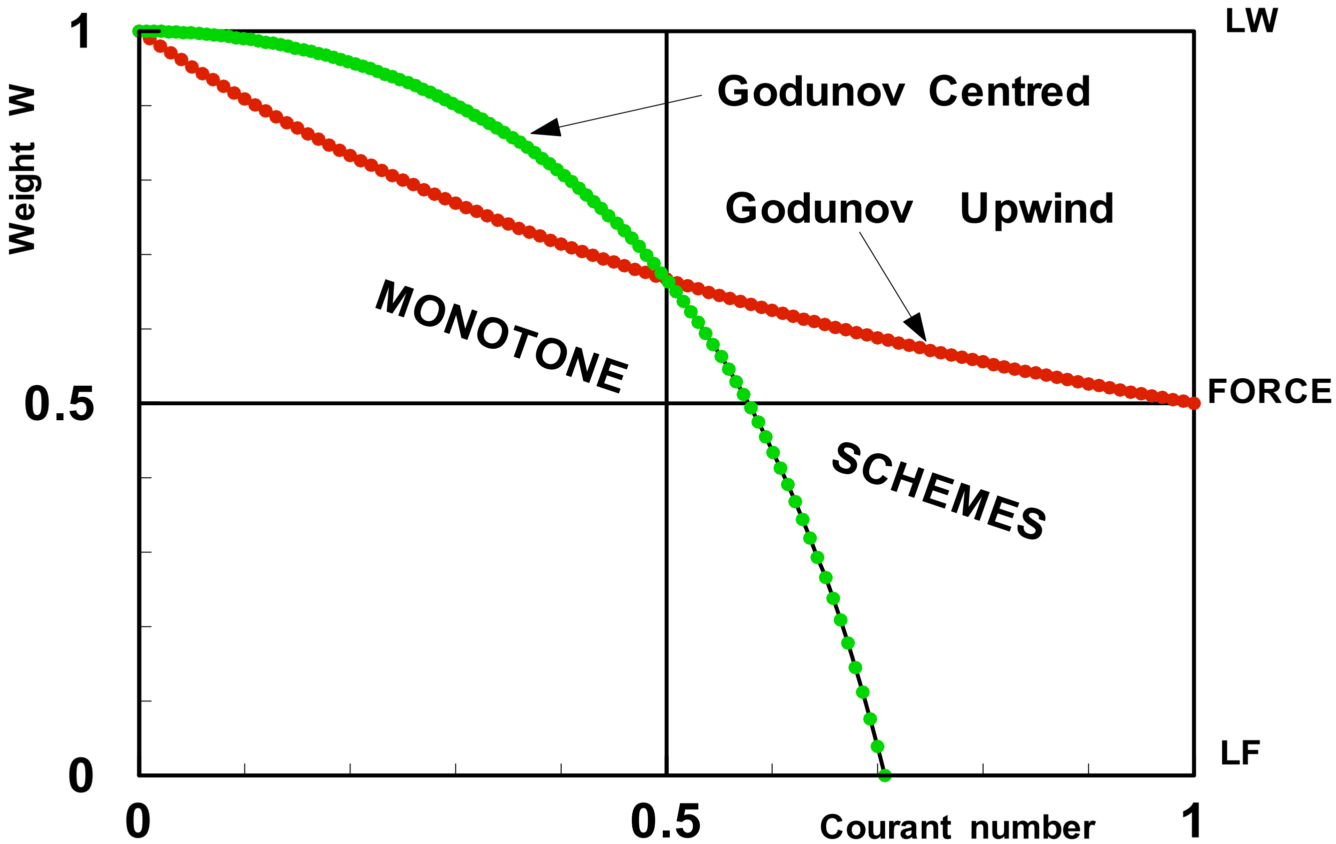
$$\omega = 1 \text{ (Lax - Wendroff)}$$

$$\omega = 1/2 \text{ (FORCE)}$$

$$\omega = \frac{1}{1+c} \text{ (GFORCE)}$$

Monotonocity

$$0 \le \omega \le \omega_{\max} \equiv \frac{1}{1+|c|} \qquad \frac{1}{2} \le \omega_{\max} \equiv \frac{1}{1+|c|} \le 1$$



FORCE's friends and relatives

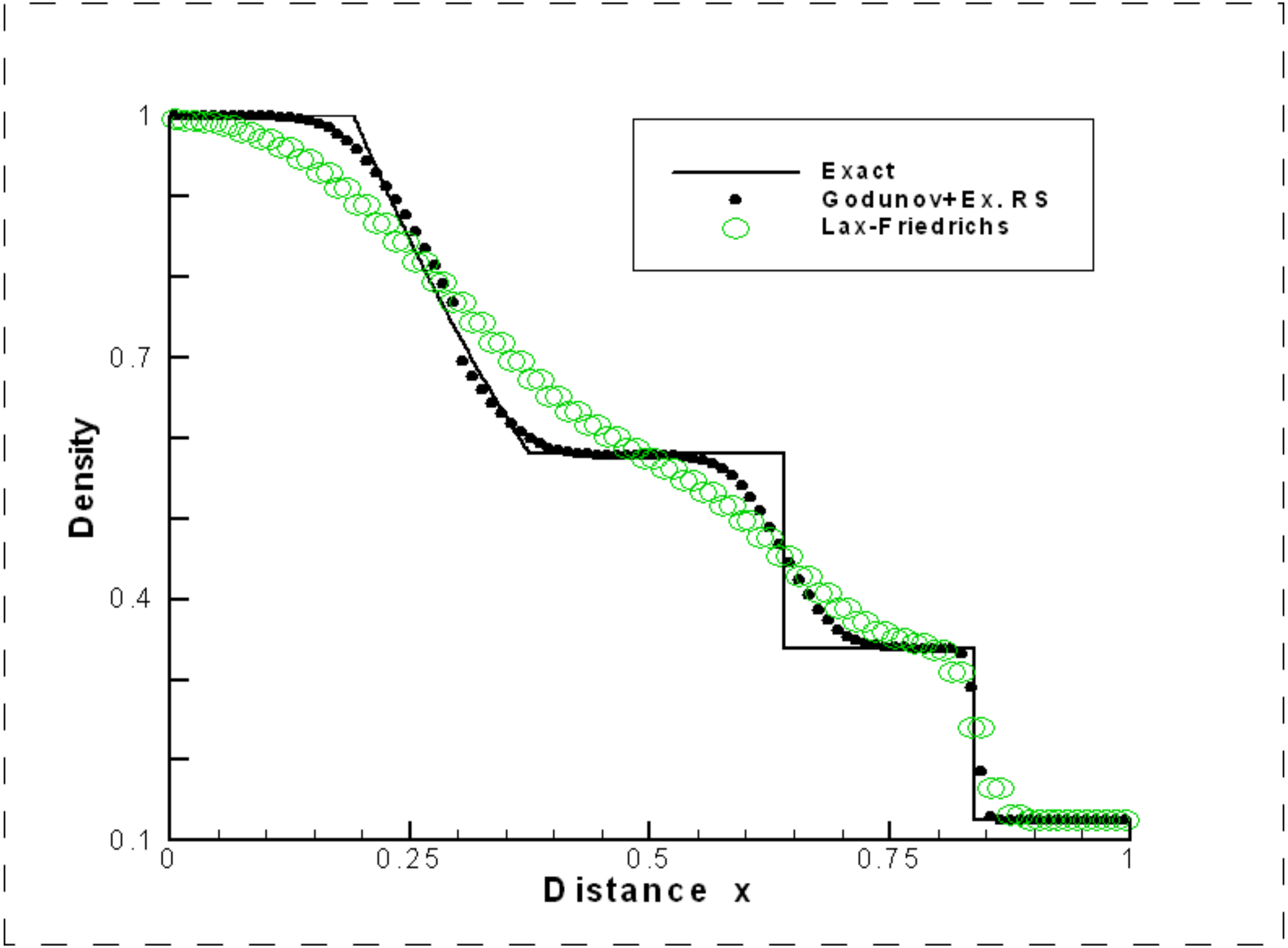
- The composite schemes of Liska and Wendroff (friend)

Liska R and Wendroff B. Composite schemes for conservation laws. SIAM J. Numerical Analysis, Vol. 35, pp 2250-2271, 1998

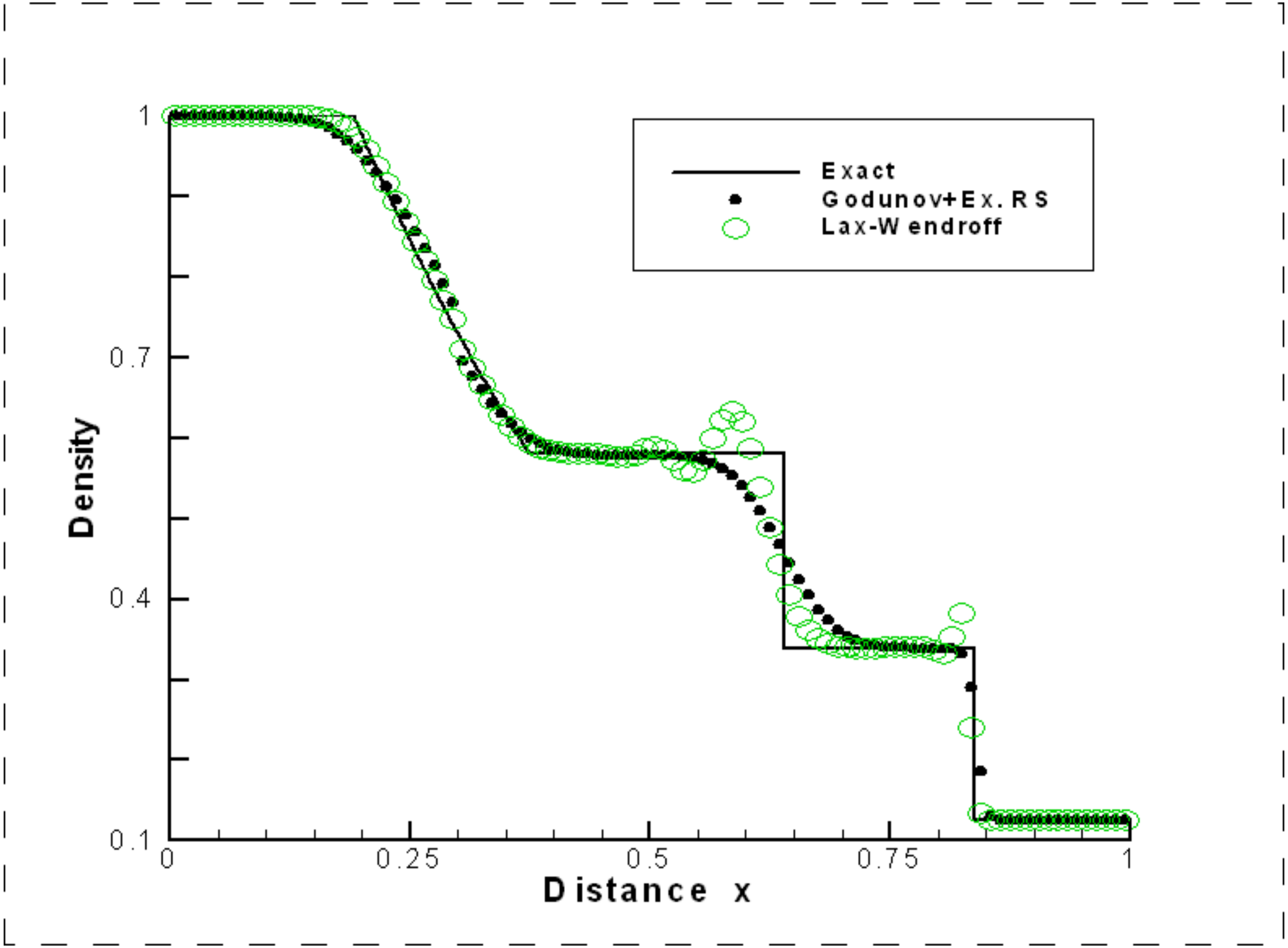
- The centred scheme of Nessyahu and Tadmor (relative)

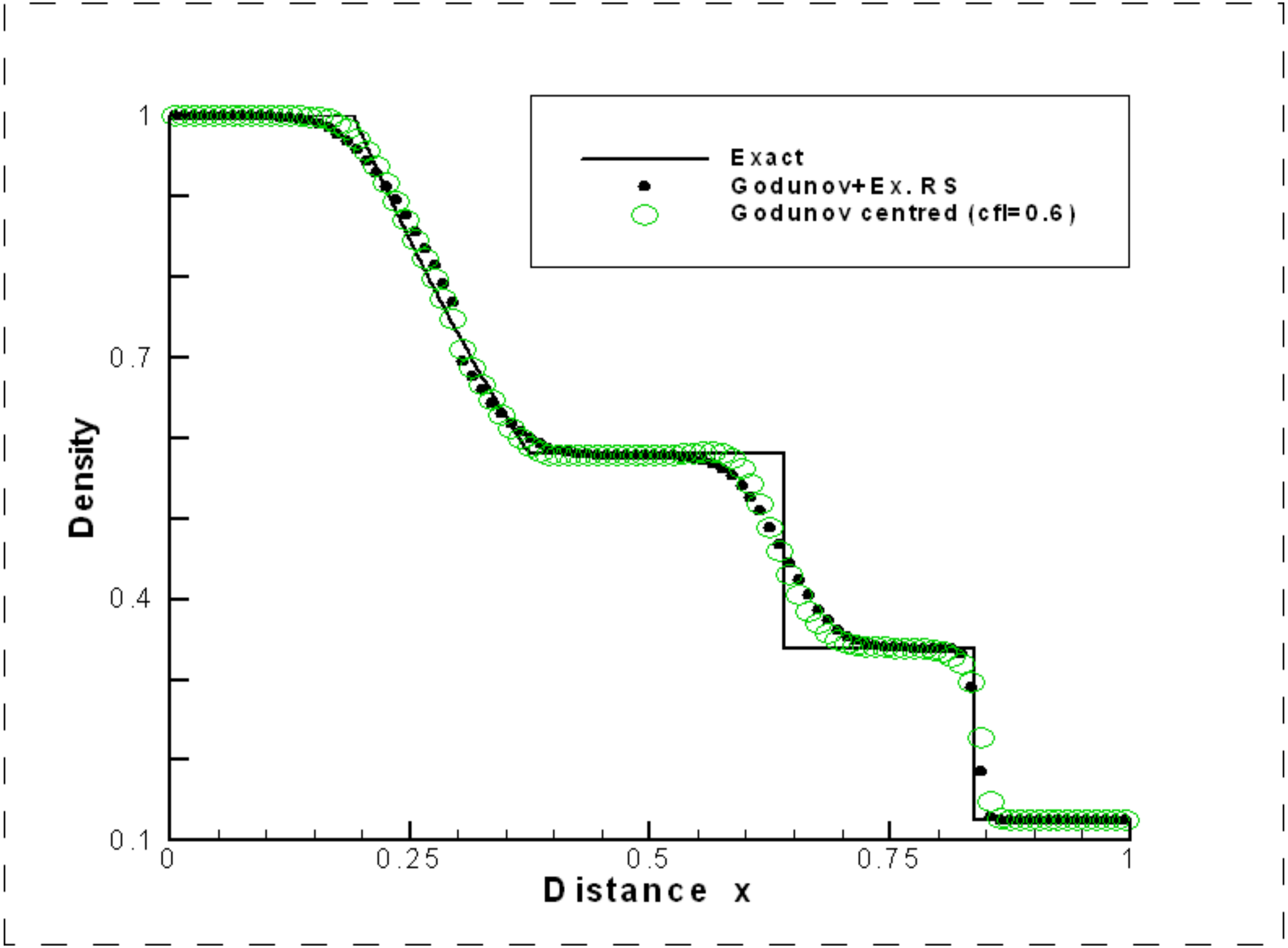
Non-oscillatory central differencing for hyperbolic conservation Laws. J. Computational Physics, Vol 87, pp 408-463, 1990.

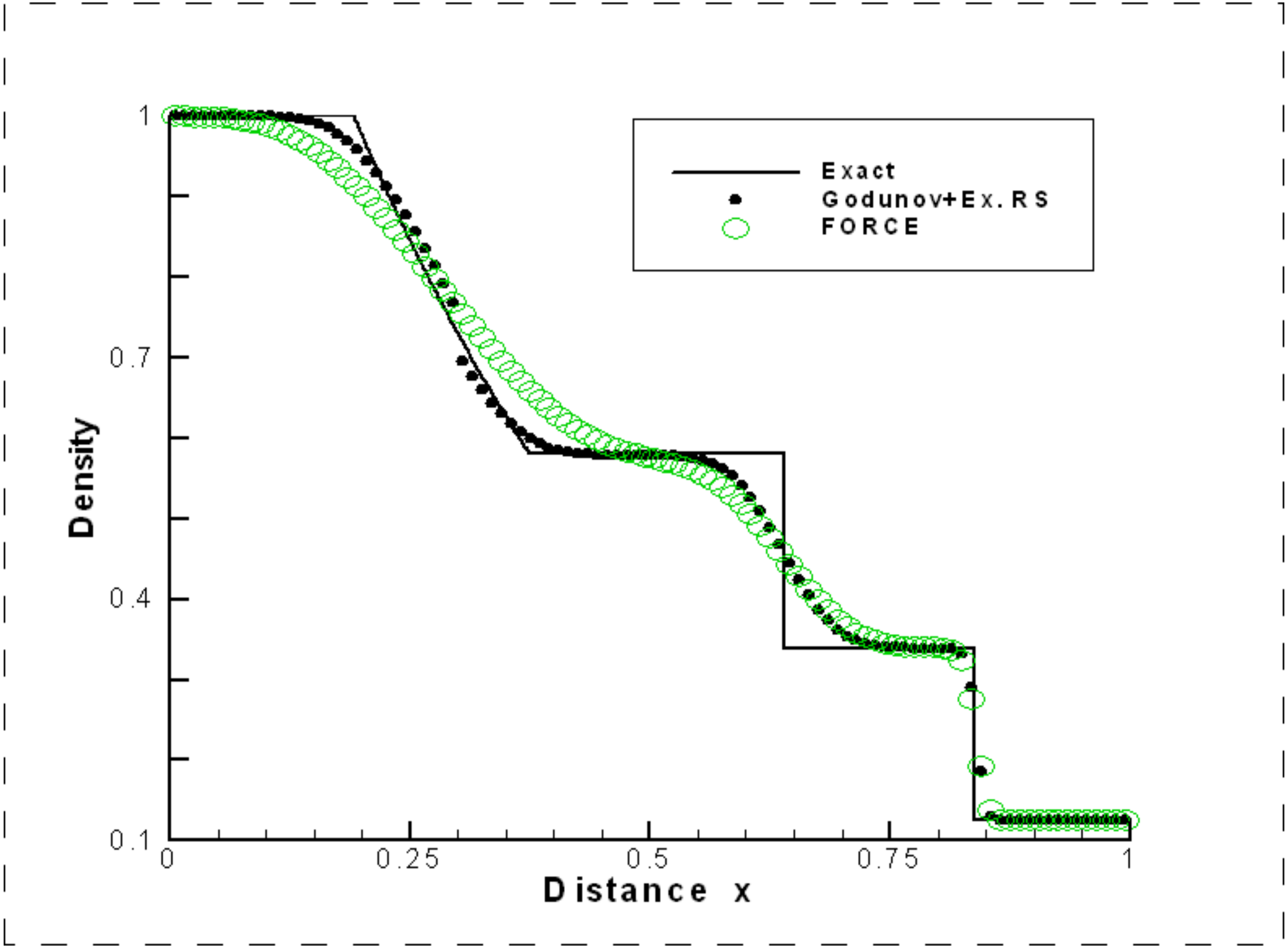
Numerical results



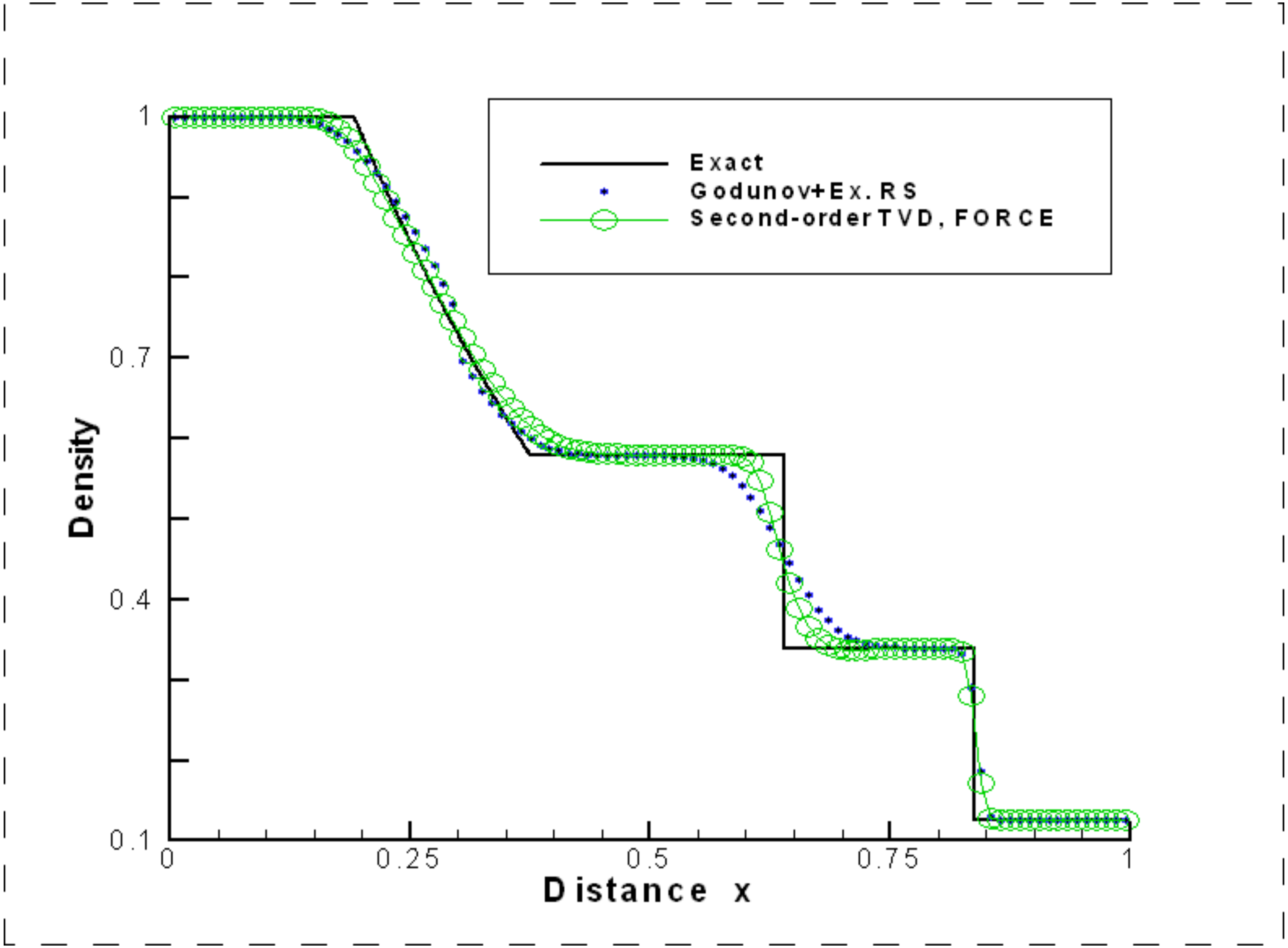
Exact
Godunov+Ex. RS
Lax-Friedrichs
Density
Distance x
1
0.7
0.4
0.1
0
0.25
0.5
0.75
1


Exact
Godunov+Ex. RS
Lax-Wendroff
Density
Distance x
1
0.7
0.4
0.1
0
0.25
0.5
0.75
1


Exact
Godunov+Ex. RS
Godunov centred (cfl=0.6)
Density
Distance x
1
0.7
0.4
0.1
0
0.25
0.5
0.75
1



Exact
Godunov+Ex. RS
FORCE
1
0.7
0.4
0.1
Density
0
0.25
0.5
0.75
1
Distance x



Exact
Godunov+Ex. RS
Second-order TVD, FORCE
1
0.7
0.4
0.1
Density
0
0.25
0.5
0.75
1
Distance x

How about extensions of FORCE ?

- **High-order non-oscillatory extensions**

- **Source terms**

- **Multiple space dimensions**

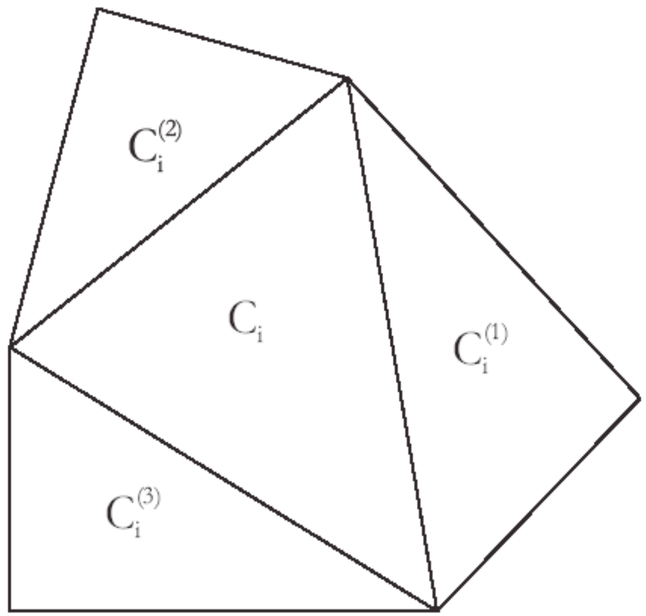
- **Unstructured meshes**

FORCE schemes on unstructured meshes

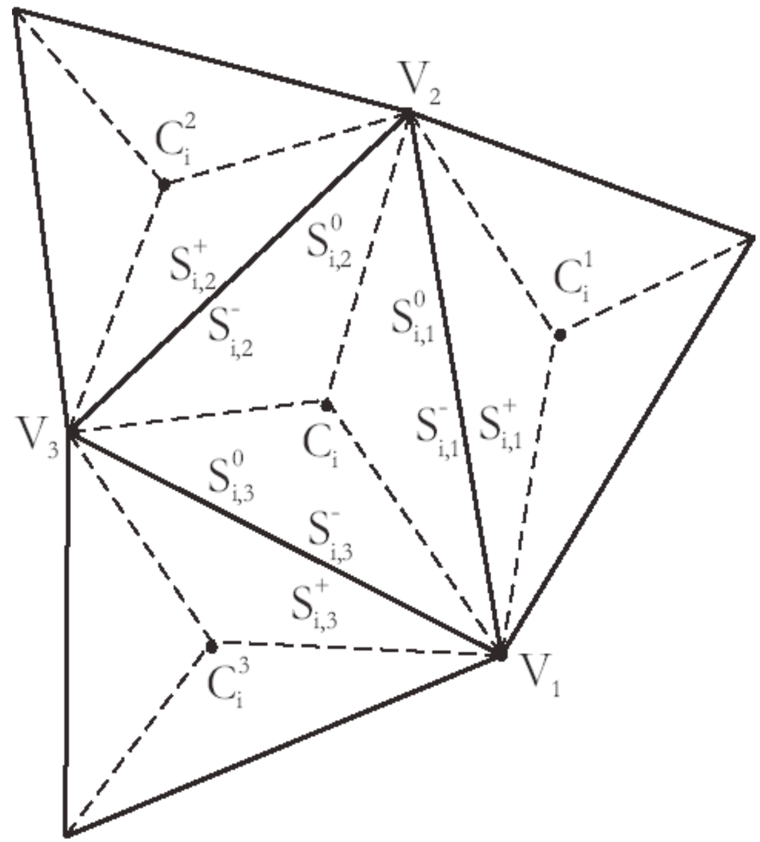
Toro E F, Hidalgo A and Dumbser M.

FORCE schemes on unstructured meshes I: Conservative hyperbolic systems.
(Journal of Computational Physics, Vol. 228, pp 3368-3389, 2009)

Illustration in 2D



Triangular primary mesh



Primary and secondary mesh

Step I

Initial condition: integral averages at time n Q_i^n

Averaging operator applied on edge-base control volume gives

$$\mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{\mathbf{Q}_i^n V_i^- + \mathbf{Q}_j^n V_j^+}{V_j^- + V_j^+} - \frac{1}{2}\frac{\Delta t S_j}{V_j^- + V_j^+}\left(\underline{\underline{\mathbf{F}}}\left(\mathbf{Q}_j^n\right) - \underline{\underline{\mathbf{F}}}\left(\mathbf{Q}_i^n\right)\right) \cdot \vec{n}_j \qquad \underline{\underline{F}} = (F, G, H)$$

V_j^- Portion of j edge-base volume inside cell i

V_j^+ Portion of j edge-base volume outside cell i

S_j Area of face j (between cells i and j)

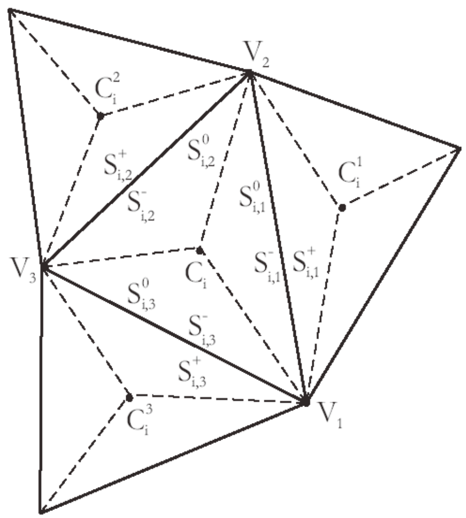
n_j Unit outward normal vector to of face j

Step II

Initial condition: integral averages at time n+1/2 $Q_{j+1/2}^{n+1/2}$

Averaging operator applied on primary mesh gives

$$\mathbf{Q}_i^{n+1} = \frac{1}{|T_i|}\sum_{j=1}^{n_f}\left(\mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}}V_j^- - \frac{1}{2}\Delta t S_j \underline{\underline{\mathbf{F}}}\left(\mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right)\cdot\vec{n}_j\right)$$



Step III: one-step conservative scheme

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{|T_i|} \sum_{j=1}^{n_f} S_j \underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{\mathrm{FORCE}\alpha} \cdot \vec{n}_j$$

$$\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{\mathrm{FORCE}\alpha} = \frac{1}{2}\left(\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LW\alpha}\left(\mathbf{Q}_i^n, \mathbf{Q}_j^n\right) + \underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LF\alpha}\left(\mathbf{Q}_i^n, \mathbf{Q}_j^n\right)\right)$$

$$\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LW\alpha}\left(\mathbf{Q}_i^n, \mathbf{Q}_j^n\right) = \underline{\underline{\mathbf{F}}}\left(\mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) \ ,$$

$$\mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{\mathbf{Q}_i^n V_i^- + \mathbf{Q}_j^n V_j^+}{V_j^- + V_j^+} - \frac{1}{2}\frac{\Delta t S_j}{V_j^- + V_j^+}\left(\underline{\underline{\mathbf{F}}}\left(\mathbf{Q}_j^n\right) - \underline{\underline{\mathbf{F}}}\left(\mathbf{Q}_i^n\right)\right) \cdot \vec{n}_j$$

$$\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LF\alpha}\left(\mathbf{Q}_i^n, \mathbf{Q}_j^n\right) = \frac{V_j^- \underline{\underline{\mathbf{F}}}\left(\mathbf{Q}_j^n\right) + V_j^+ \underline{\underline{\mathbf{F}}}\left(\mathbf{Q}_i^n\right)}{V_j^- + V_j^+} - \frac{V_j^- V_j^+}{V_j^- + V_j^+}\frac{2}{\Delta t S_j}\left(\mathbf{Q}_j^n - \mathbf{Q}_i^n\right)\vec{n}_j^T$$

The FORCE flux in α space dimensions on Cartesian meshes

$$F^{force\alpha}_{i+1/2,j,k} = \frac{1}{2}\left(F^{lw\alpha}_{i+1/2,j,k} + F^{lf\alpha}_{i+1/2,j,k}\right)$$

Lax-Wendroff type flux

$$F^{lw\alpha}_{i+1/2,j} = F(Q^{lw\alpha}_{i+1/2,j}),$$

$$Q^{lw\alpha}_{i+1/2,j,k} = \frac{1}{2}\left(Q^{n}_{i,j,k} + Q^{n}_{i+1,j,k}\right) - \frac{1}{2}\frac{\alpha\Delta t}{\Delta x}\left(F(Q^{n}_{i+1,j,k}) - F(Q^{n}_{i,j,k})\right)$$

Lax-Friedrichs type flux

$$F^{lf\alpha}_{i+1/2,j,k} = \frac{1}{2}\left(F(Q^{n}_{i,j,k}) + F(Q^{n}_{i+1,j,k})\right) - \frac{1}{2}\frac{\Delta x}{\alpha\Delta t}\left(Q^{n}_{i+1,j,k} - Q^{n}_{i,j,k}\right)$$

FORCE-type fluxes

$$\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{\mathrm{GFORCE}\alpha} = \omega\,\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LW\alpha}\left(\mathbf{Q}_i^n, \mathbf{Q}_j^n\right) + (1-\omega)\,\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LF\alpha}\left(\mathbf{Q}_i^n, \mathbf{Q}_j^n\right)$$

Stability and monotonicity results

ω	$1D$	$2D$	$3D$
$0 \le \omega < \frac{1}{2}$	$\lvert c\rvert \le \frac{1}{\alpha}$	$\lvert c_x\rvert, \lvert c_y\rvert \le \frac{1}{2}$	$\lvert c_x\rvert, \lvert c_y\rvert, \lvert c_z\rvert \le \frac{1}{3}$
$\omega = \frac{1}{2}$	$\lvert c\rvert \le \frac{\sqrt{2\alpha-1}}{\alpha}$	$c_x^2 + c_y^2 \le \frac{1}{2}$	$c_x^2 + c_y^2 + c_z^2 \le \frac{1}{3}$
$\frac{1}{2} < \omega < 1$	$\lvert c\rvert \le \left\lvert\frac{-1+\omega}{\omega\alpha}\right\rvert$	$\lvert c_x\rvert, \lvert c_y\rvert \le \left\lvert\frac{-1+\omega}{2\omega}\right\rvert$	$\lvert c_x\rvert, \lvert c_y\rvert, \lvert c_z\rvert \le \left\lvert\frac{-1+\omega}{3\omega}\right\rvert$

One-dimensional interpretation

$$\partial_t Q + \partial_x F(Q) = 0$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$

$$F_{i+1/2}^{force\alpha} = \frac{1}{2}(F_{i+1/2}^{lw\alpha} + F_{i+1/2}^{lf\alpha})$$

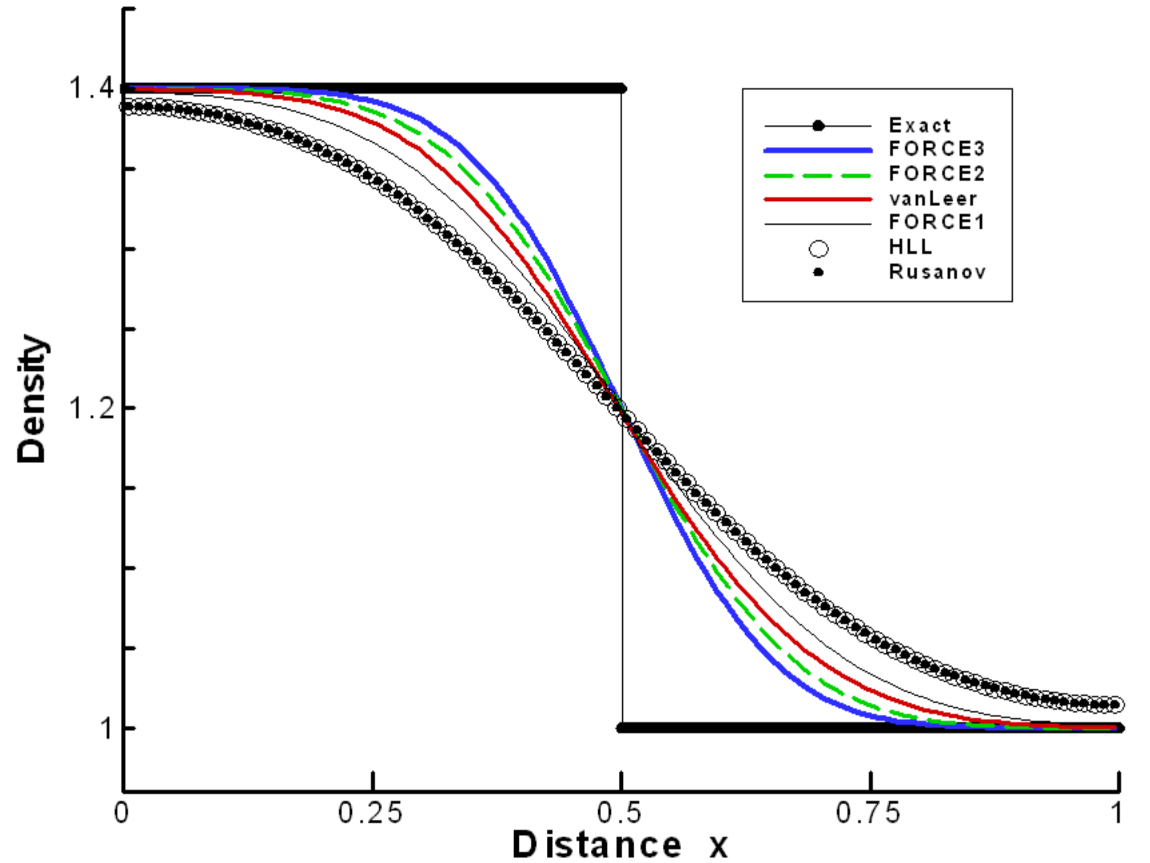
$$F_{i+1/2}^{lw\alpha} = F(Q_{i+1/2}^{lw\alpha})$$

$$Q_{i+1/2}^{lw\alpha} = \frac{1}{2}\left(Q_i^n + Q_{i+1}^n\right) - \frac{1}{2}\frac{\alpha\Delta t}{\Delta x}\left(F(Q_{i+1}^n) - F(Q_i^n)\right)$$

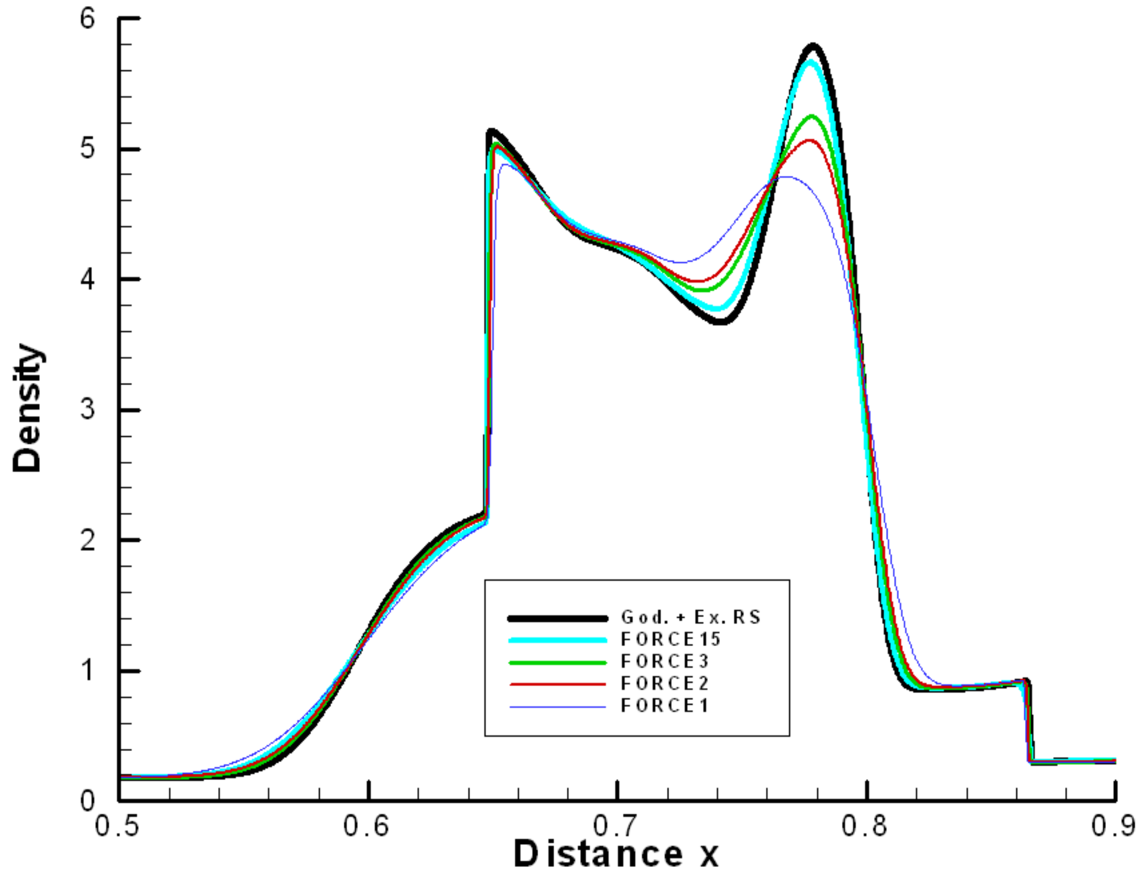
$$F_{i+1/2}^{lf\alpha} = \frac{1}{2}\left(F(Q_i^n) + F(Q_{i+1}^n)\right) - \frac{1}{2}\frac{\Delta x}{\alpha\Delta t}\left(Q_{i+1}^n - Q_i^n\right)$$

α: parameter

Numerical results for the 1D Euler equations



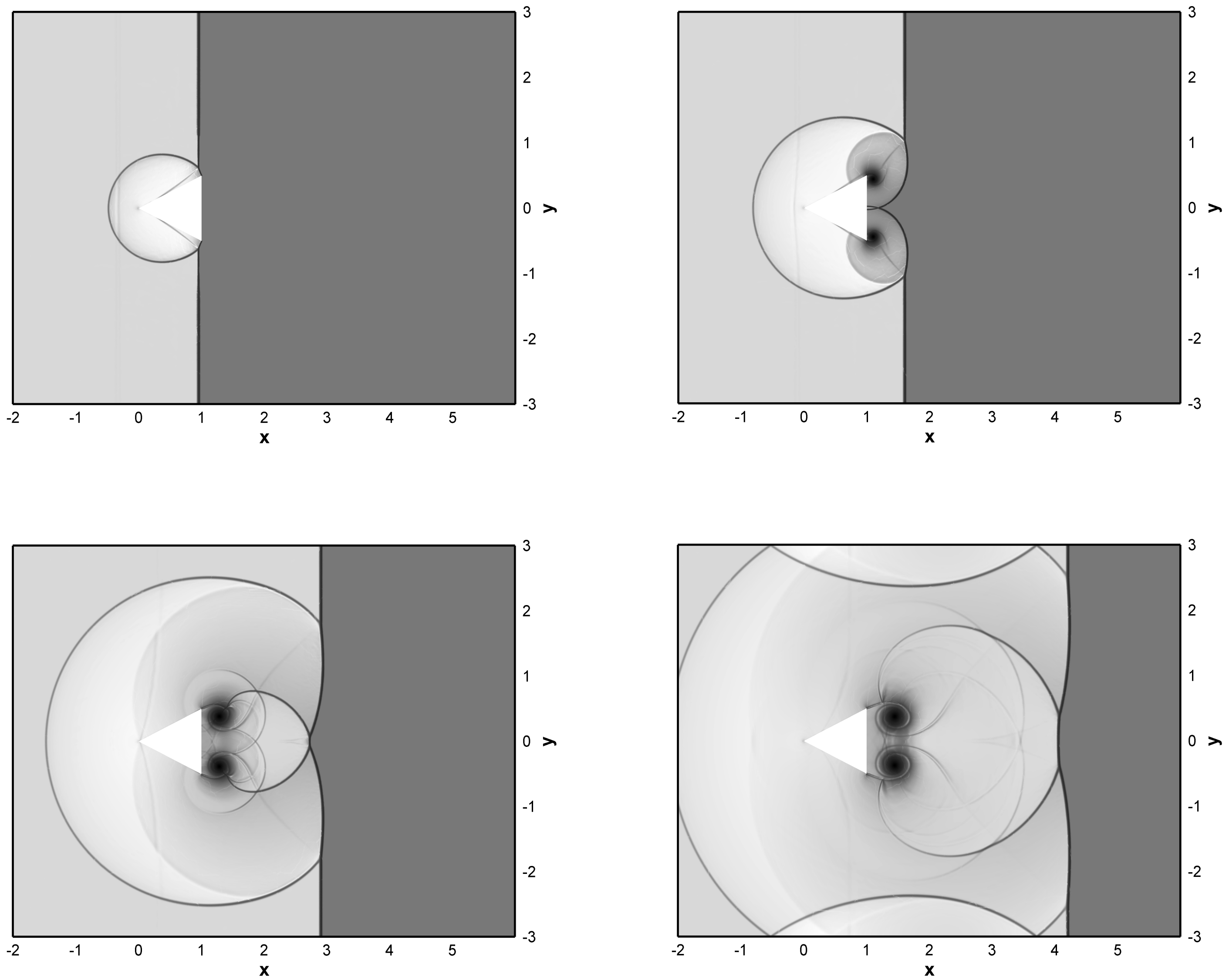
Numerical results for the 1D Euler equations



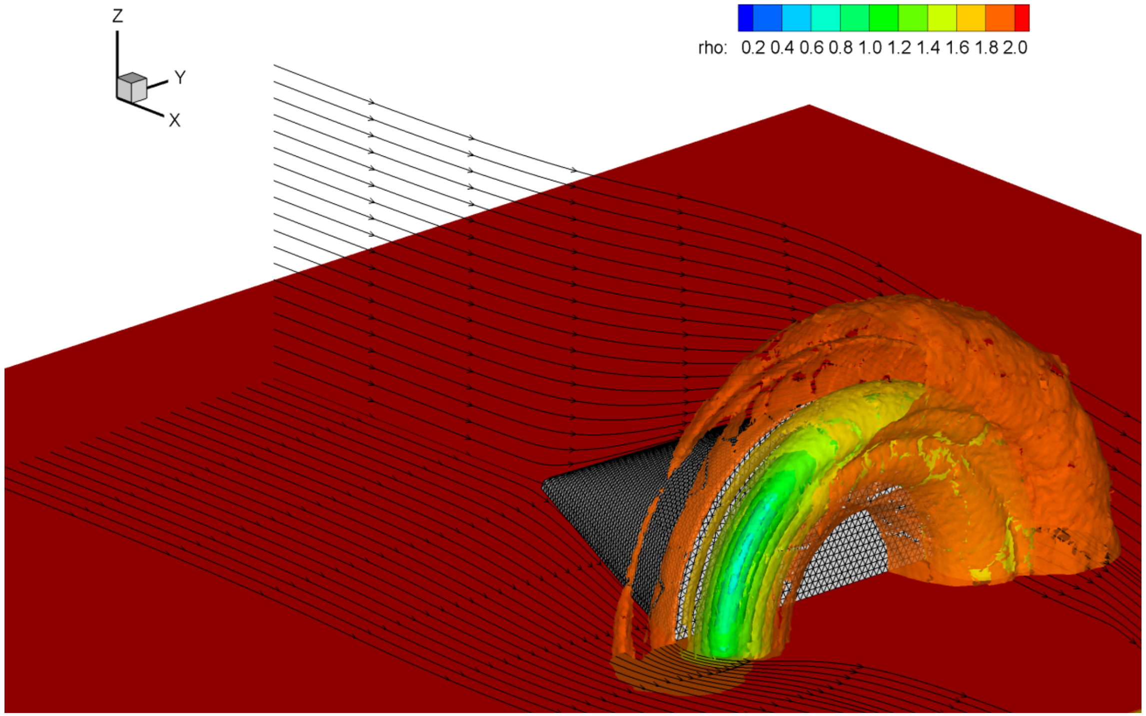
Numerical results:

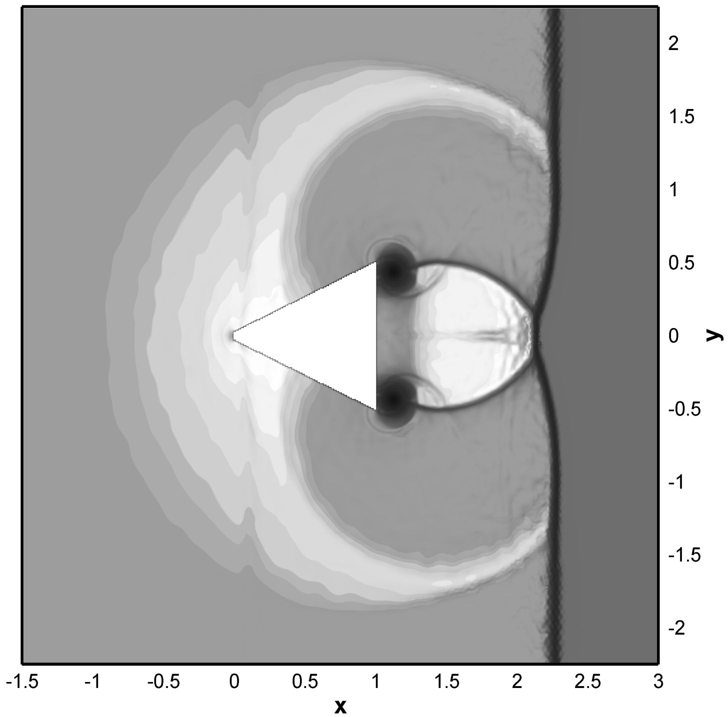
Euler equations in 2D and 3D

2D Euler equations: reflection from triangle



3D Euler equations: reflection from cone





Numerical results:

The Baer-Nunziato equations in 2D and 3D

Application of ADER to the 3D Baer-Nunziato equations

$$
\left.
\begin{array}{l}
\frac{\partial}{\partial t}\left(\phi_1\rho_1\right) + \nabla\cdot\left(\phi_1\rho_1\mathbf{u_1}\right) = 0, \\
\frac{\partial}{\partial t}\left(\phi_1\rho_1\mathbf{u_1}\right) + \nabla\cdot\left(\phi_1\rho_1\mathbf{u_1}\otimes\mathbf{u_1}\right) + \nabla\phi_1 p_1 = p_I\nabla\phi_1 + \lambda\left(\mathbf{u_2}-\mathbf{u_1}\right), \\
\frac{\partial}{\partial t}\left(\phi_1\rho_1 E_1\right) + \nabla\cdot\left(\left(\phi_1\rho_1 E_1 + \phi_1 p_1\right)\mathbf{u_1}\right) = -p_I\partial_t\phi_1 + \lambda\,\mathbf{u_I}\cdot\left(\mathbf{u_2}-\mathbf{u_1}\right), \\
\frac{\partial}{\partial t}\left(\phi_2\rho_2\right) + \nabla\cdot\left(\phi_2\rho_2\mathbf{u_2}\right) = 0, \\
\frac{\partial}{\partial t}\left(\phi_2\rho_2\mathbf{u_2}\right) + \nabla\cdot\left(\phi_2\rho_2\mathbf{u_2}\otimes\mathbf{u_2}\right) + \nabla\phi_2 p_2 = p_I\nabla\phi_2 - \lambda\left(\mathbf{u_2}-\mathbf{u_1}\right), \\
\frac{\partial}{\partial t}\left(\phi_2\rho_2 E_2\right) + \nabla\cdot\left(\left(\phi_2\rho_2 E_2 + \phi_2 p_2\right)\mathbf{u_2}\right) = p_I\partial_t\phi_1 - \lambda\,\mathbf{u_I}\cdot\left(\mathbf{u_2}-\mathbf{u_1}\right), \\
\frac{\partial}{\partial t}\phi_1 + \mathbf{u_I}\nabla\phi_1 = 0.
\end{array}
\right\}
\tag{54}
$$

11 non-linear hyperbolic PDES
stiff source terms: relaxation terms

EXTENSION TO NONCONSERVATIVE SYSTEMS:
Path-conservative schemes

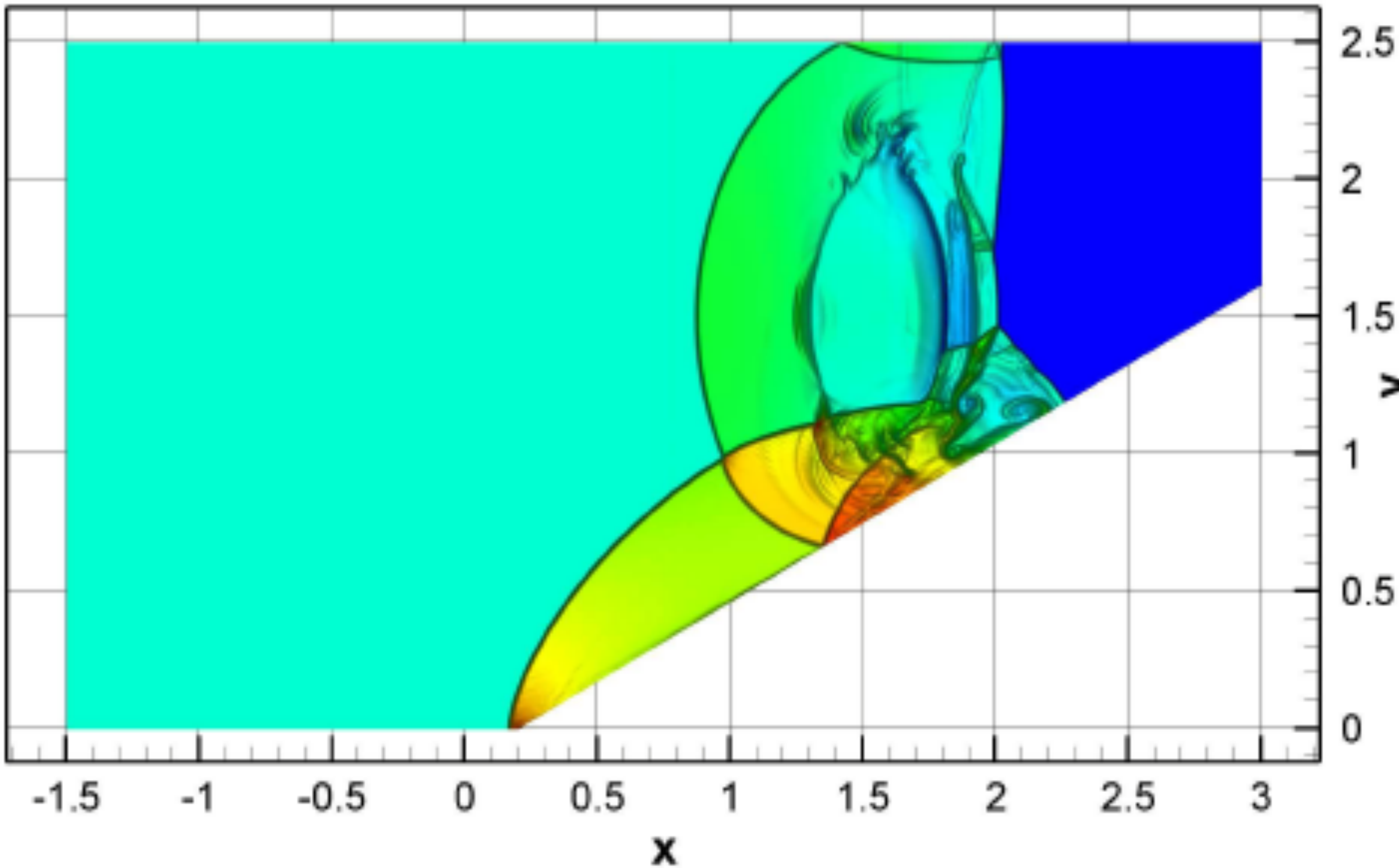
DUMBSER M, HIDALGO A, CASTRO M, PARES C, TORO E F.

FORCE schemes on unstructured meshes II:
Nonconservative hyperbolic systems.
Computer Methods in Applied Science and Engineering. Online version available, 2010
Also published (NI09005-NPA) in pre-print series of the
Newton Institute for Mathematical Sciences
University of Cambridge, UK.

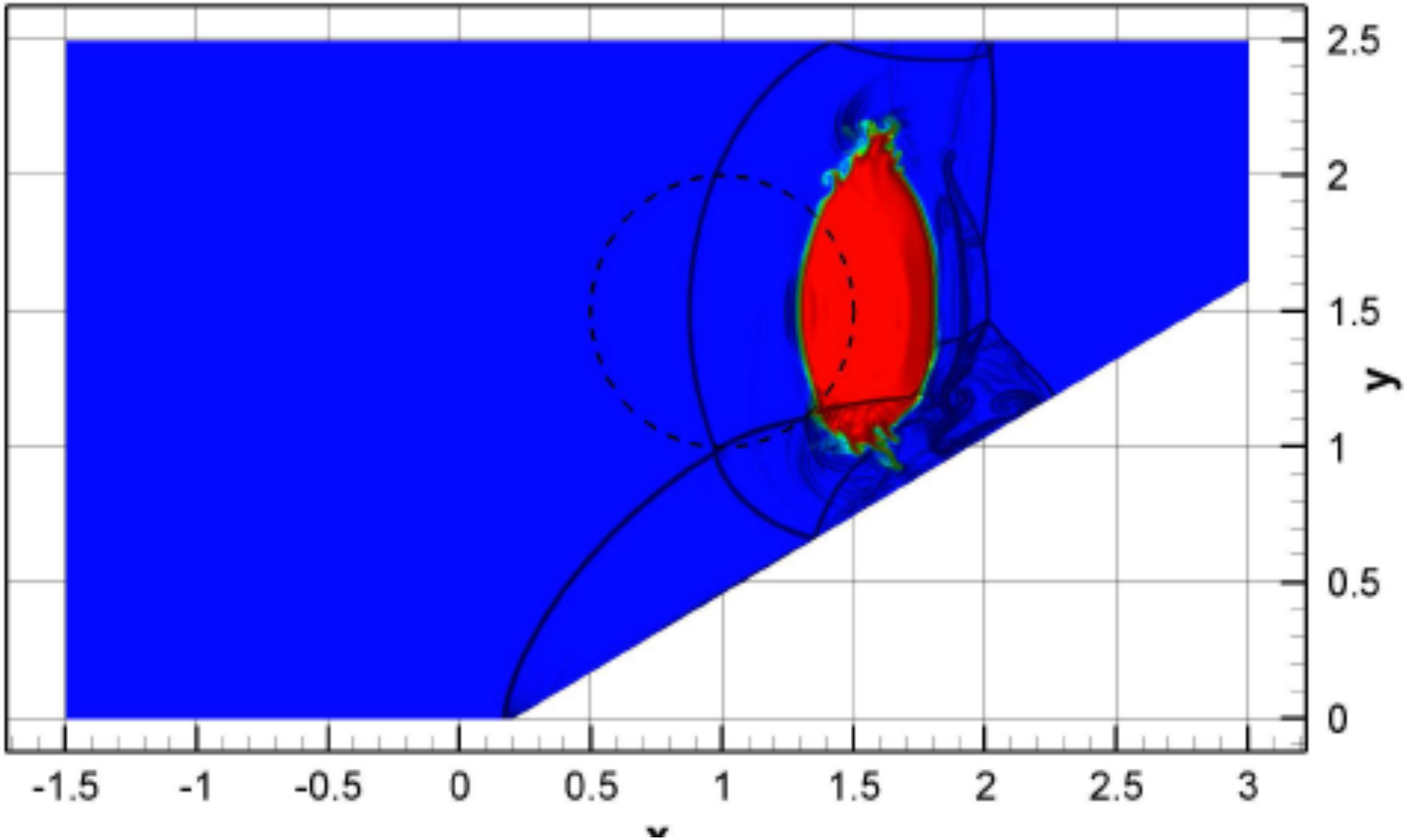
It can be downloaded from
http://www.newton.ac.uk/preprints2009.html

CASTRO M, PARDO A, PARES C, TORO E F. (2009).
ON SOME FAST WELL-BALANCED FIRST ORDER SOLVERS FOR NONCONSERVATIVE SYSTEMS.
MATHEMATICS OF COMPUTATION. ISSN: 0025-5718. Accepted.

Double Mach reflection for the 2D Baer-Nunziato equations



Double Mach reflection for the 2D Baer-Nunziato equations



Summary on FORCE

- ***A centred scheme***
- ***One-step scheme***
- ***In conservative form, with a numerical flux***
- ***Monotone***
- ***Linearly stable up to CFL =1, 1/2, 1/3***
- ***Very simple to use, applicable to any system (useful for complicated systems)***
- ***High-order extensions (TVD, WENO, DG, ADER)***

- Further reading: Chapters 18 of:

Toro E F. Riemann solvers and numerical methods for fluid dynamics. Springer, Third Edition, 2009.