HLLC Riemann solver

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HLLC, as HLL, assumes a wave model.

- For the 1D (also 3D+species equations) Euler equations the assumption of a 3-wave model is exact (complete Riemann solver).

- For the 2D shallow water equations (also with species equations) a 3-wave model is exact.

- HLLC has been applied to systems with more than 3 distinct characteristic fields (MHD equations), selecting the waves to be included in the model (not complete).

- HLLC has also been applied to the two-phase Baer-Nunziato equations (complete). To be summarized in this lecture.

Recalling the Godunov Scheme

$$\partial_t Q + \partial_x F(Q) = 0$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$

To compute $F_{i+1/2}^{GOD}$ we must solve the Riemann problem

$$\left.\begin{aligned} &\partial_t Q + \partial_x F(Q) = 0 \\ &Q(x,0) = \begin{cases} Q_i^n \ , x < x_{i+1/2} \\ Q_{i+1}^n \ , x > x_{i+1/2} \end{cases} \end{aligned}\right\} \rightarrow \quad \mathrm{Q}_{\mathrm{i+1/2}}(\mathrm{x}/\mathrm{t})$$

$$F_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} F(Q_{i+1/2}(0))dt = F(Q_{i+1/2}(0))$$

3D case in normal direction

$$Q_t + F(Q)_x = 0$$

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix} \qquad F(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(E + p) \end{bmatrix}$$

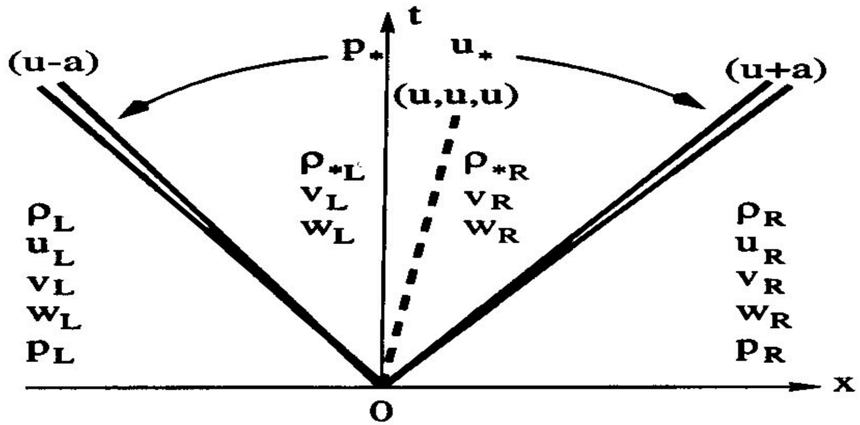
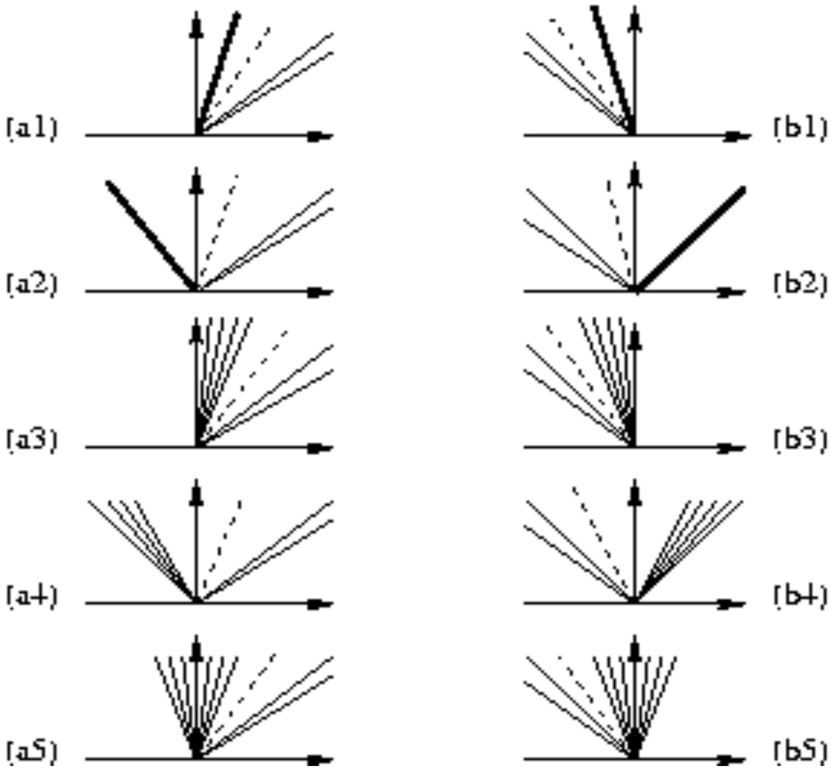


Fig. 4.18. Structure of the solution of the Riemann problem for the split three-dimensional case

In general, for the 1D Euler equations, there are 10 possible wave configurations to consider in the solution sampling. See Fig. below.

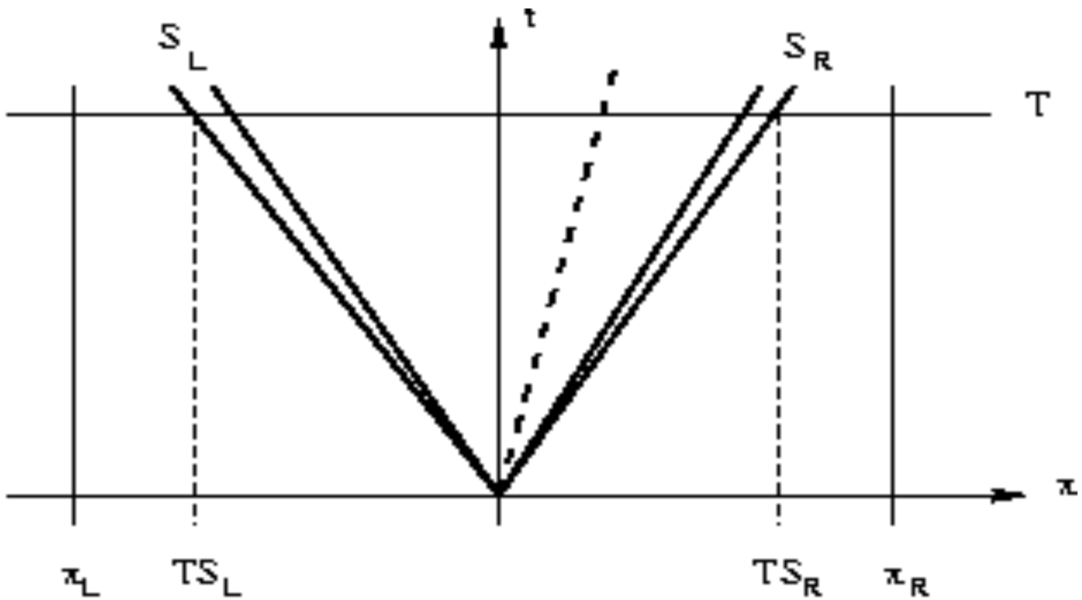


The Harten-Lax-van Leer approach (HLL) 1983

The HLL (Harten-Lax-van Leer) Riemann solver

(A Harten, P Lax and B van Leer. On upstream differencing and Godunov type methods for hyperbolic conservation laws. SIAM review. 25(1), pp 35-61, 1983)

It is assumed a solution structure that only includes the fastest waves

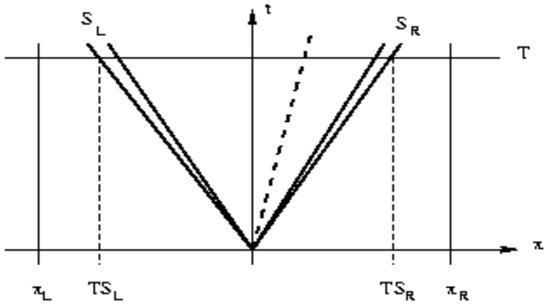


Moreover, it is assumed that estimates for these wave speeds are available

Construction of the HLL flux

1. Apply integral form of the conservation laws in volume:

$$[TS_L, TS_R] \times [0, T]$$



$$\frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} Q(x,T)\,dx = \frac{S_R Q_R - S_L Q_L + F_L - F_R}{S_R - S_L} = Q^{hll}$$

2. Apply integral form of the conservation laws in volume:

$$[TS_L, 0] \times [0, T]$$

$$\int_{TS_L}^{0} Q(x,T)\,dx = \int_{TS_L}^{0} Q(x,0)\,dx + \int_0^T F_L\,dt - \int_0^T F^{hll}\,dt \rightarrow$$

$$F^{hll} = F_L - S_L Q_L - \frac{1}{T} \int_{TS_L}^{0} Q(x,T)\,dx$$

3. Substitute Q^{hll} into last integral

$$F_0 = F_L - S_L Q_L - \frac{1}{T}\int_{TS_L}^{0} Q(x,T)\,dx$$

$$F_0 = F_L - S_L Q_L - \frac{1}{T}\int_{TS_L}^{0} \mathrm{Q}^{\mathrm{hll}}\,dx$$

$$F_0 = F_L + S_L(\mathrm{Q}^{\mathrm{hll}} - Q_L)$$

4. Algebraic manipulations give the HLL flux along interface

$$F^{hll} = F_L + S_L(\mathrm{Q}^{\mathrm{hll}} - Q_L)$$

$$F^{hll} = \frac{S_R F_L - S_L F_R + S_L S_R (Q_R - Q_L)}{S_R - S_L}$$

$$F_{i+1/2}^{hll} = \begin{cases} F_L & \text{if} \quad 0 \le S_R \\ F^{hll} = \frac{S_R F_L - S_L F_R + S_R S_L (Q_R - Q_L)}{S_R - S_L} & , \; S_L \le 0 \le S_R \\ F_R & \text{if} \quad 0 \ge S_R \end{cases}$$

Rusanov’s flux (1961)

Consider the HLL flux

$$F^{hll} = \frac{S_R F_L - S_L F_R + S_R S_L (Q_R - Q_L)}{S_R - S_L}$$

Two wave speed estimates are needed: $S_L \;;\; S_R$

Assume a single wave speed estimate: $S_+ > 0$

Define a second speed: $S_L = -S_+ \;;\; S_R = +S_+$

Substitution into HLL flux gives the Rusanov flux

$$F^{Rusavov} = \tfrac{1}{2}(F_L + F_R) - \tfrac{S_+}{2}(Q_R - Q_L)$$

This flux is sometimes called (wrongly in my view) the Local Lax-Friedrichs or simply the Lax-Friedrichs flux.

Note that

if in the Rusanov flux

$$F^{Rusavov} = \tfrac{1}{2}(F_L + F_R) - \tfrac{S_+}{2}(Q_R - Q_L)$$

$$S_+ = \frac{\Delta x}{\Delta t}$$

we then reproduce the Lax-Friedrichs flux.

$$F^{Lax-F} = \tfrac{1}{2}(F_L + F_R) - \tfrac{1}{2}\frac{\Delta x}{\Delta t}(Q_R - Q_L)$$

In this sense the (centred) Lax-Friedrichs flux can be seen as un upwind flux (the limiting case).

Remarks on the HLL Riemann solver.

- This Riemann solver is very simple and entropy satisfying; it performs well at critical (sonic) rarefactions. No entropy fix needed

- But note that middle waves are ignored. This results in excessive smearing of contact waves and vortices.

- Wave speed estimates are still needed, for which knowledge of the solution is required in advance. Details on schemes to provide wave speed estimates given later

A weak feature of HLL:

absence of intermediate waves:

In particular:

- Entropy waves
- Slip surfaces
- Material interfaces
- Vortical flows
- Ignition fronts
- Shear layers
- Contact discontinuities

HLL (left) HLLC (right)

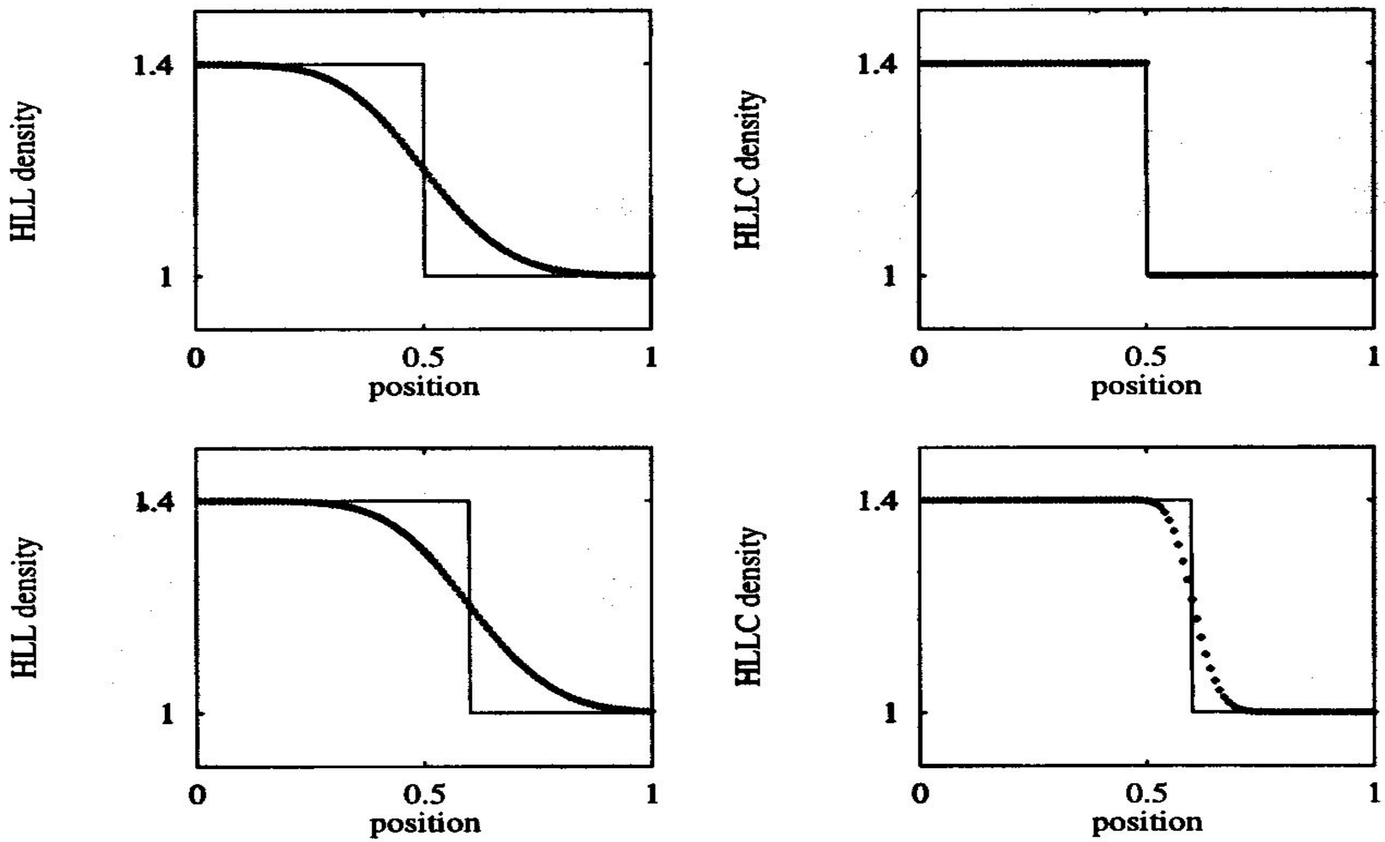


Fig. 10.20. Godunov's method with HLL (left) and HLLC (right) Riemann solvers applied to Tests 6 and 7. Numerical (symbol) and exact (line) solutions are compared at time 2.0

Futher reading on the HLL Riemann solver

Harten A, Lax P and van Leer B. On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. SIAM Review, Vol. 25, pp:35-61, 1983

Toro E F. Riemann solvers and numerical methods for fluid dynamics. Springer, Third Edition, 2010. Chapter 10.

..............................and references therein...

The HLLC Riemann solver
Toro et al. (1992, 1994)

A quick search with google gave me:

HLLC: Healesville Living and Learning Centre

HLLC: Happy Land Learning Center

HLLC: House of Lords Liaison Committee

HLLC: Home Loan Learning Center

HLLC: Harten, Lax, van Leer and (the missing) Contact

The HLLC solver (Toro et al. 1992, 1994) is a modification of the HLL Riemann solve

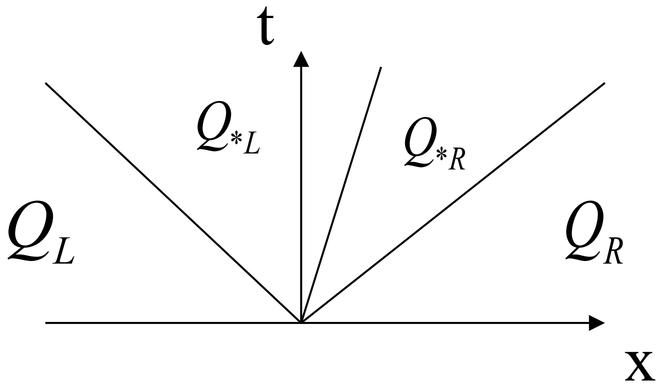
C stands for CONTACT

The contact wave is included in the structure of the solution of the Riemann problem

Now the Star Region has two sub-regions (for a 3 by 3 system)

Further developments on HLLC:
Toro and Chakraborty, 1994
Batten et al. 1997a, 1997b

The HLLC Riemann Solver (cont….)



$$F_{i+1/2}^{hllc} = \begin{cases} F_L \, , 0 \le S_L \\ F_{*L} \, , \, S_L \le 0 \le S_* \\ F_{*R} \, , \, S_* \le 0 \le S_R \\ F_R \, , 0 \ge S_R \end{cases}$$

$$F_{*L} = F_L + S_L(Q_{*L} - Q_L) \, , \qquad F_{*R} = F_R + S_R(Q_{*R} - Q_R)$$

We have 4 unknown vectors:

$$Q_{*L}, \quad Q_{*R}, \quad F_{*L}, \quad F_{*R}$$

First solve for the states: $Q_{*L}, \quad Q_{*R}$

Then solve for the fluxes: $F_{*L}, \quad F_{*R}$

We assume the following conditions in the star region:

$$\left.\begin{aligned}
p_{*L} &= p_{*R} \equiv p_* \\
u_{*L} &= u_{*R} \equiv u_* \equiv S_* \\
v_{*L} &= v_L \\
v_{*R} &= v_R \\
w_{*L} &= w_L \\
w_{*R} &= w_R
\end{aligned}\right\}$$

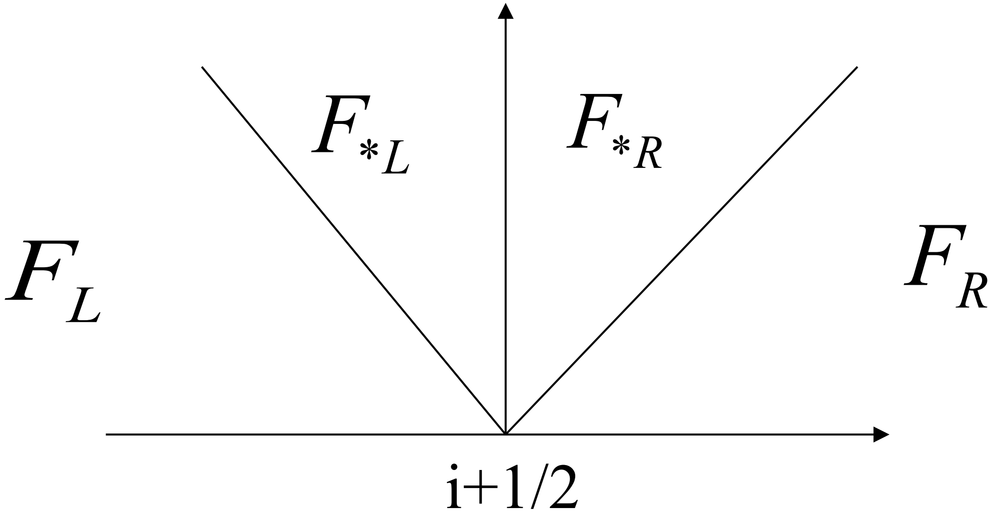
These conditions are satisfied by the exact solution. See Toro 2010 (Springer).

Then we can write

$$S_L Q_{*L} - F_{*L} = M_L(Q_L, Q_R)$$
$$S_R Q_{*R} - F_{*R} = M_R(Q_L, Q_R)$$

where the right hand sides are known functions.

Algebraic manipulations give the solution for the unknown states, from which the sought flux vectors follow.



$$F_{*L} = F_L + S_L(Q_{*L} - Q_L)\ , \qquad F_{*R} = F_R + S_R(Q_{*R} - Q_R)$$

$$Q_{*K} = \rho_K\left(\frac{S_K - u_K}{S_K - S_*}\right)\begin{bmatrix} 1 \\ s_* \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_* - u_K)\left[S_* + \frac{p_K}{\rho_K(S_K - u_K)}\right] \end{bmatrix} \qquad K = L, R$$

The 3D multi-component case

$$Q_t + F(Q)_x = 0$$

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \\ \rho\phi_1 \\ \rho\phi_2 \\ \cdots\cdot \\ \rho\phi_m \end{bmatrix} \qquad F(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(E+p) \\ u\rho\phi_1 \\ u\rho\phi_1 \\ \cdots\cdot \\ u\rho\phi_m \end{bmatrix}$$

Obtain eigenvalue u of multiplicity m+3 and the HLLC procedure goes through.

Wave-Speed Estimates for HLLC

We need estimates S_L, S_*, S_R

Find estimates for p_*, u_*

$$S_L = u_L - a_L q_L\,,\; S_* = u_*\;,\; S_R = u_R + a_R q_R$$

$$q_K = \begin{cases} 1 & p_* \le p_K \quad \text{rarefaction} \\ \sqrt{1 + \frac{\gamma+1}{2\gamma}\left(\frac{p_*}{p_K} - 1\right)} & p_* > p_K \quad \text{shock} \end{cases}$$

Pressure-velocity estimates.

Primitive-Variable Riemann Solver

• Linearize the non-conservative system:

$$W_t + A(W)W_x = 0$$

by 'freezing' coefficient matrix at a state $\hat{W}$

• Standard linear theory gives the explicit solution:

$$p_* = \tfrac{1}{2}(p_L + p_R) + \tfrac{1}{2}(u_L - u_R)(\hat{\rho}\hat{a})$$
$$u_* = \tfrac{1}{2}(u_L + u_R) + \tfrac{1}{2}(p_L - p_R)/(\hat{\rho}\hat{a})$$

HLLC--summary

$$p_* = \tfrac{1}{2}(p_L + p_R) + \tfrac{1}{2}(u_L - u_R)(\hat{\rho}\hat{a})$$

$$u_* = \tfrac{1}{2}(u_L + u_R) + \tfrac{1}{2}(p_L - p_R)/(\hat{\rho}\hat{a}) \qquad q_K = \begin{cases} 1 & p_* \le p_K \\ \sqrt{1 + \frac{\gamma+1}{2\gamma}(\frac{p_*}{p_K} - 1)} & p_* > p_K \end{cases}$$

$$\hat{\rho} = \tfrac{1}{2}(\rho_L + \rho_R) \qquad \hat{a} = \tfrac{1}{2}(a_L + a_R)$$

$$S_L = u_L - a_L q_L, \; S_* = u_*, \; S_R = u_R + a_R q_R$$

$$F_{i+1/2}^{hllc} = \begin{cases} F_L, \; 0 \le S_L \\ F_{*L}, \; S_L \le 0 \le S_* \\ F_{*R}, \; S_* \le 0 \le S_R \\ F_R, \; 0 \ge S_R \end{cases} \qquad F_{*L} = F_L + S_L(Q_{*L} - Q_L), \qquad F_{*R} = F_R + S_R(Q_{*R} - Q_R)$$

$$Q_{*K} = \rho_K\left(\frac{S_K - u_K}{S_K - S_*}\right)\begin{bmatrix} 1 \\ s_* \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_* - u_K)[S_* + \frac{p_K}{\rho_K(S_K - u_K)}] \end{bmatrix} \qquad K = L, R$$

HLL versus HLLC

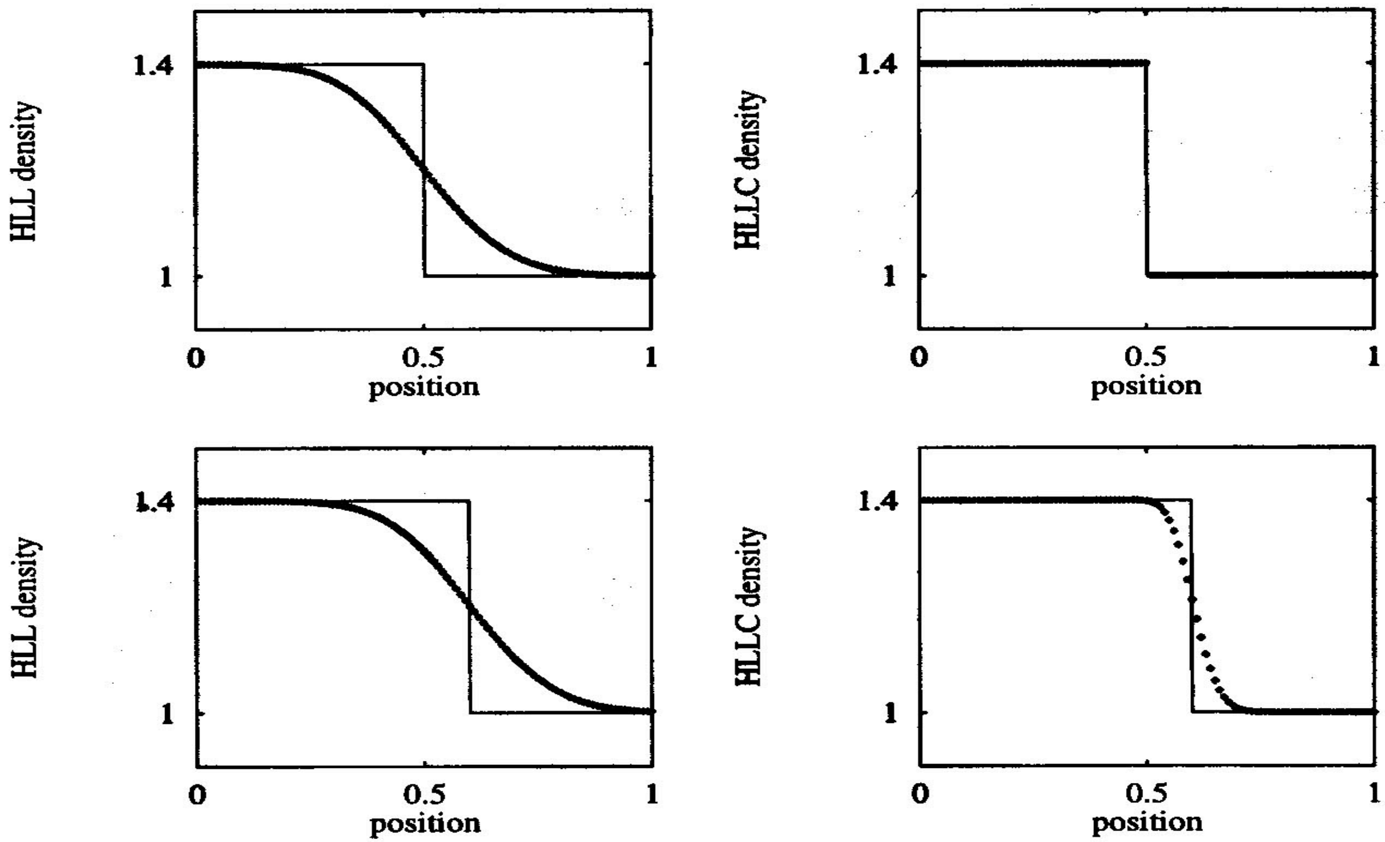


Fig. 10.20. Godunov's method with HLL (left) and HLLC (right) Riemann solvers applied to Tests 6 and 7. Numerical (symbol) and exact (line) solutions are compared at time 2.0

Extensions and Applications of HLLC

- Shallow water equations (Toro, 2001)
- MHD (K F Gurski, SIAM J Sc Comput., 2004)
- MHD (Shengtai Li, JCP)
- Relativistic MHD: Mignone, Massaglia and Bodo
- High-order extensions via:
  the WAF method (Toro, 1989)
  ADER method (Toro et al. 2001 and others)
  WENO methods (Titarev and Toro)
  Discontinuous Galerkin Finite Element Methods (van der Vegt, 2002)

- 2D multiphase flows (Toro, 1992)
- Implicit version for compressible turbulent flows (Batten et al. 1997)
- Multiphase, multi-dimensional flows (Toro 1992, Saurel, 2002)
- …. and many more, including packages and comercial software.

  http://vulcan-cfd.larc.nasa.gov/index.html

Futher reading on the HLLC Riemann solver

Toro E F, Spruce M and Spears W. Restoration of the contact surface in the HLL Riemann solver. Shock Waves, Vol. 4, pp: 25-34, 1994.

(Also as Cranfield University Technical Report, 1992)

Toro E F and Chakraborty. Development of an approximate Riemann solver for the steady supersonic Euler equations. The Aeronautical Journal, Vol. 98, pp: 325-339, 1994.

Batten P, Leschziner M and Goldberg U C. Average sate jacobians and implicit methods for viscous and turbulen flows. J. Comput. Phys. Vol. 137, pp: 38-78, 1997.

Toro E F. Riemann solvers and numerical methods for fluid dynamics. Springer, Third edition, 2010. Chapter 10.

The Rusanov Riemann solver (1961)

and

The Lax-Friedrichs flux (1960)

HLLC applied to the shallow water equations

$$\partial_t Q + \partial_x F(Q) + \partial_y G(Q) = 0$$

$$Q = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} \; ; \; F(Q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} g h^2 \\ huv \end{bmatrix} \; ; \; G(Q) = \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2} g h^2 \end{bmatrix}$$

Augmented 1D problem normal to interface

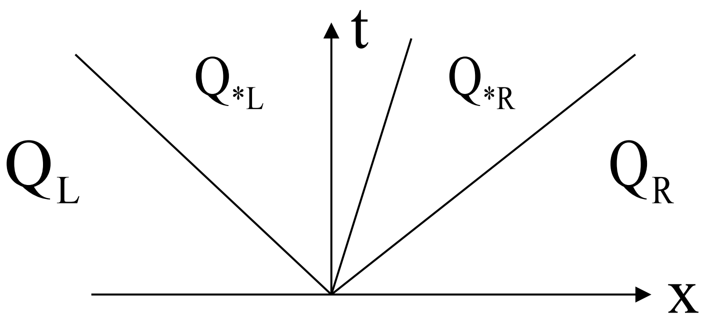
$$\partial_t Q + \partial_x F(Q) = 0$$

$$Q = \begin{bmatrix} h \\ hu \\ h\psi \end{bmatrix} \; ; \; F(Q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} g h^2 \\ hu\psi \end{bmatrix}$$

The HLLC Riemann solver

The HLLC is a modification of the HLL Riemann solver; C stands for Contact.

- Contact and shear waves, missing in HLL, are included in the structure the solution of the Riemann problem
- Now the Star Region has two sub-regions (for a 3 by 3 system)



$$F_{i+1/2}^{hllc} = \begin{cases} F_L \,, 0 \le S_L \\ F_{*L} \,, \; S_L \le 0 \le S_* \\ F_{*R} \,, \; S_* \le 0 \le S_R \\ F_R \,, 0 \ge S_R \end{cases}$$

$$F_{*L} = F_L + S_L(Q_{*L} - Q_L) \,, \qquad F_{*R} = F_R + S_R(Q_{*R} - Q_R)$$

$$S_L, \; S_*, \; S_R$$

are wave speed estimates

Then the star states are

$$Q_{*K} = h_K \left(\frac{S_K - u_K}{S_K - S_*}\right) \begin{bmatrix} 1 \\ S_* \\ \psi_K \end{bmatrix} \quad K = L, R$$

and the HLLC intercell flux is

$$F_{i+1/2}^{hllc} = \begin{cases} F_L, \; 0 \le S_L \\ F_{*L}, \; S_L \le 0 \le S_* \\ F_{*R}, \; S_* \le 0 \le S_R \\ F_R, \; 0 \ge S_R \end{cases}$$

Wave speed estimates for HLL and HLLC

We need estimates S_L, S_*, S_R

We could use the eigenvalues:

$$\lambda_1 = u - a, \quad \lambda_2 = u, \quad \lambda_3 = u + a$$

This is NOT recommended.

We could use information from other Riemann solvers.

For example, we could use the Roe average eigenvalues
As recommended by Einfeldt (1988). This works well.

Wave speed estimates based on depth and particle velocity in the STAR region

$$h_*, u_*$$

Then we set:

$$S_L = u_L - a_L q_L \ , \ S_* = u_* \ , S_R = u_R + a_R q_R$$

$$q_K = \begin{cases} 1 & , h_* \le h_K \quad \text{Rarefaction} \\ \sqrt{\frac{1}{2}\left[\frac{(h_* + h_K)h_*}{h_K^2}\right]} & , h_* > h_K \quad \text{Shock} \end{cases}$$

This choice of speed is in a sense exact. Exact wave relations have been used

Use Depth-positive values h_*, u_* for example

The Depth-Positive Riemann Solver

$$h_* = \tfrac{1}{2}(h_L + h_R) - \tfrac{1}{4}(u_R - u_L)(\tfrac{h_L + h_R}{a_L + a_R})$$

$$u_* = \tfrac{1}{2}(u_L + u_R) - \tfrac{1}{4}(h_R - h_L)(\tfrac{a_L + a_R}{h_L + h_R})$$

This approximate Riemann solver has the same depth-positivity condition as the exact solver.

For details see E F Toro. Shock-capturing methods for free-surface shallow flows, John Wiley and Sons, 2003, chapter 10.

HLLC applied to the Baer-Nunziato equations

SA Tokareva and E F Toro. HLLC-type Riemann Solver for the Baer-Nunziato Equations of Compressible Two-Phase Flow. Journal of Computational Physics. (to appear, 2010)

The Baer-Nunziato equations

$$\partial_t Q + \partial_x F(Q) + T(Q)\partial_x \overline{\alpha} = S(Q)$$

$$Q = \begin{bmatrix} \overline{\alpha}\overline{\rho} \\ \overline{\alpha}\overline{\rho}\overline{u} \\ \overline{\alpha}\overline{\rho}\overline{E} \\ \overline{\alpha} \\ \hat{\alpha}\hat{\rho} \\ \hat{\alpha}\hat{\rho}\hat{u} \\ \hat{\alpha}\hat{\rho}\hat{E} \end{bmatrix} \quad F(Q) = \begin{bmatrix} \overline{\alpha}\overline{\rho}\overline{u} \\ \overline{\alpha}\overline{\rho}\overline{u}^2 + \overline{\alpha}\overline{p} \\ \overline{u}(\overline{\alpha}\overline{\rho}\overline{E} + \overline{\alpha}\overline{p}) \\ 0 \\ \hat{\alpha}\hat{\rho}\hat{u} \\ \hat{\alpha}\hat{\rho}\hat{u}^2 + \hat{\alpha}\hat{p} \\ \hat{u}(\hat{\alpha}\hat{\rho}\hat{E} + \hat{\alpha}\hat{p}) \end{bmatrix} \quad T(Q) = \begin{bmatrix} 0 \\ -P_i \\ -P_i V_i \\ V_i \\ 0 \\ P_i \\ P_i V_i \end{bmatrix} \quad S(Q) = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{bmatrix}$$

$\overline{Q}$: Solid phase $\hat{Q}$: Gas phase

$$V_i = \overline{u} \qquad \mathrm{P_i} = \hat{p}$$

First published comprehensive mathematical analysis of the equations due to:

Embid P and Baer M. Continuum Mechanics and Thermodynamics, Vol. 4 (1992), pp 279-312

Analysis and INDIRECT Riemann solver due to:

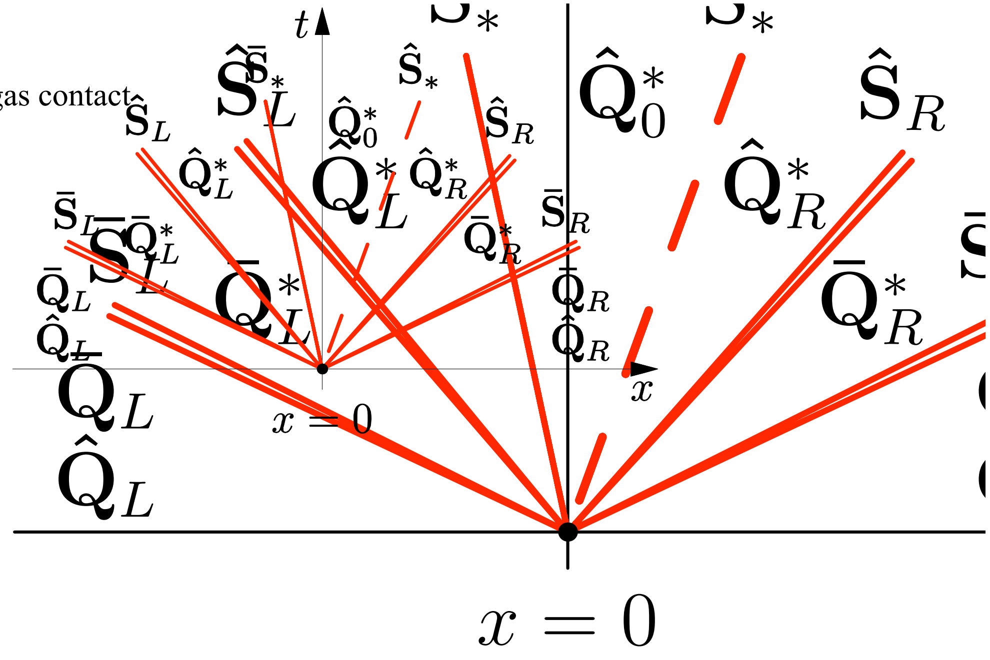
Andrianov N and Warnecke G. The Riemann problem for the Baer-Nunziato Two-Phase Flow Model. Journal of Computational Physics, Vol 195, pp 434-464, 2004.

First published DIRECT Riemann solver due to:

Schwendemann D W, Wahle C W and Kapila A K. Journal of Computational Physics, Vol. 212, pp 490-526, 2006.

The Riemann problem

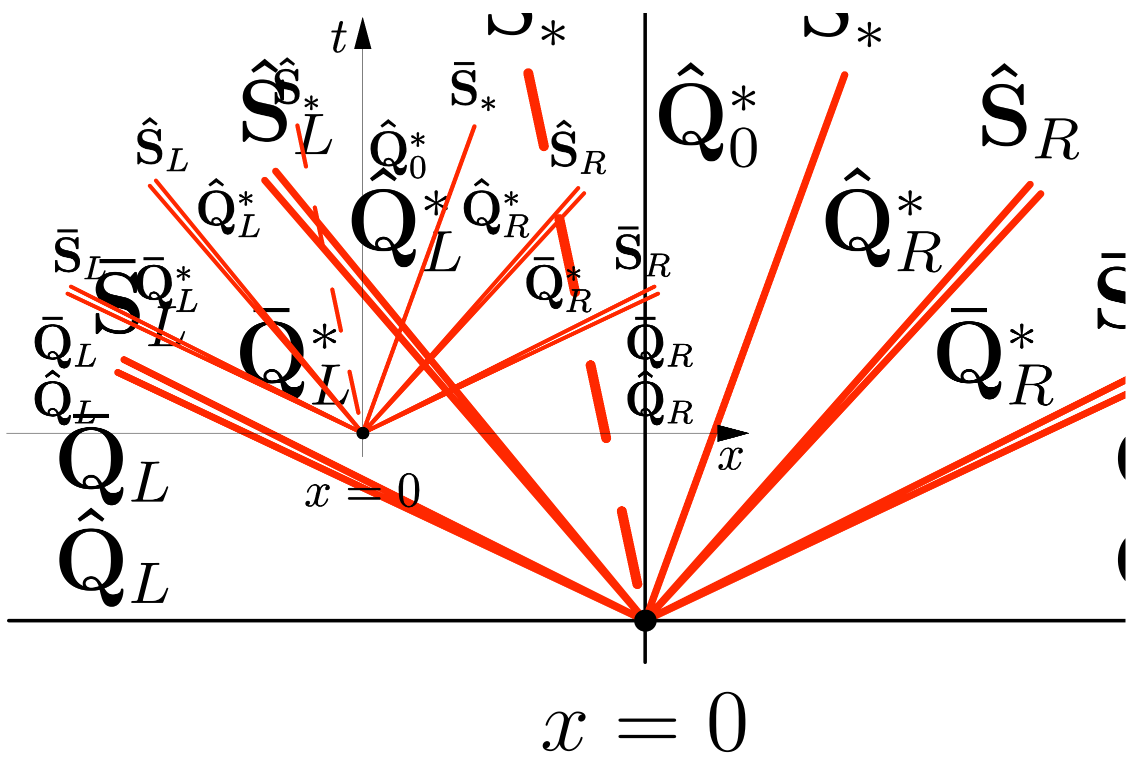
Solid contact on the left of gas contact



Across solid "contact"

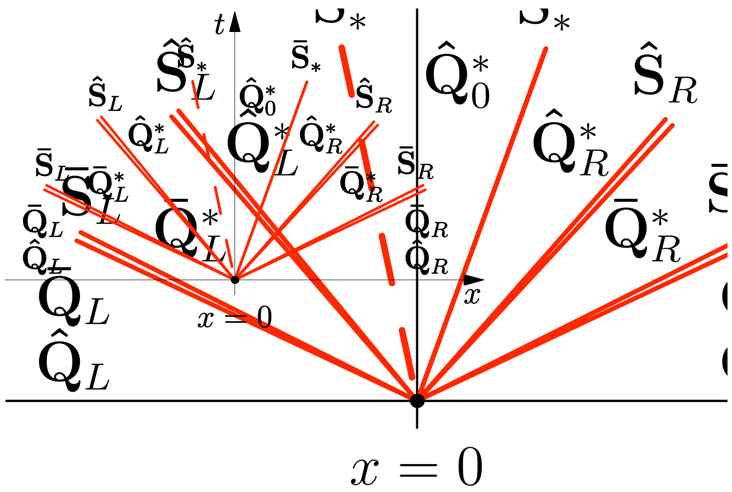
$$W = \begin{bmatrix} \bar{\rho} \\ \bar{u} \\ \bar{p} \\ \bar{\alpha} \\ \hat{\rho} \\ \hat{u} \\ \hat{p} \end{bmatrix} \qquad \Delta W = \begin{bmatrix} \Delta\bar{\rho} \neq 0 \\ \Delta\bar{u} = 0 \\ \Delta\bar{p} \neq 0 \\ \Delta\bar{\alpha} \neq 0 \\ \Delta\hat{\rho} \neq 0 \\ \Delta\hat{u} \neq 0 \\ \Delta\hat{p} \neq 0 \end{bmatrix}$$

The Riemann problem



Solid contact on the right of gas contact

Use HLLC approach to connect data states to *star* states



$$\bar{F}_{*L} = \bar{F}_L + \bar{S}_L(\bar{Q}_{*L} - \bar{Q}_L) \quad ; \quad \bar{F}_{*R} = \bar{F}_R + \bar{S}_R(\bar{Q}_{*R} - \bar{Q}_R)$$

$$\hat{F}_{*L} = \hat{F}_L + \hat{S}_L(\hat{Q}_{*L} - \hat{Q}_L) \quad ; \quad \hat{F}_{*R} = \hat{F}_R + \hat{S}_R(\hat{Q}_{*R} - \hat{Q}_R)$$

From HLLC equations for the solid phase

$$\overline{F}_{*L} = \overline{F}_L + \overline{S}_L(\overline{Q}_{*L} - \overline{Q}_L) \quad ; \quad \overline{F}_{*R} = \overline{F}_R + \overline{S}_R(\overline{Q}_{*R} - \overline{Q}_R)$$

$$\overline{\rho}_{*L}(\overline{u}) = \overline{\rho}_L\left(\frac{\overline{S}_L - \overline{u}_L}{\overline{S}_L - \overline{u}}\right)$$

$$\overline{p}_{*L}(\overline{u}) = \overline{p}_L + \overline{\rho}_L(\overline{S}_L - \overline{u}_L)(\overline{u} - \overline{u}_L)$$

$$\overline{\rho}_{*R}(\overline{u}) = \overline{\rho}_R\left(\frac{\overline{S}_R - \overline{u}_R}{\overline{S}_R - \overline{u}}\right)$$

$$\overline{p}_{*R}(\overline{u}) = \overline{p}_R + \overline{\rho}_R(\overline{S}_R - \overline{u}_R)(\overline{u} - \overline{u}_R)$$

From HLLC equations for the gas phase

$$\hat{F}_{*L} = \hat{F}_L + \hat{S}_L(\hat{Q}_{*L} - \hat{Q}_L) \quad ; \quad \hat{F}_{*R} = \hat{F}_R + \hat{S}_R(\hat{Q}_{*R} - \hat{Q}_R)$$

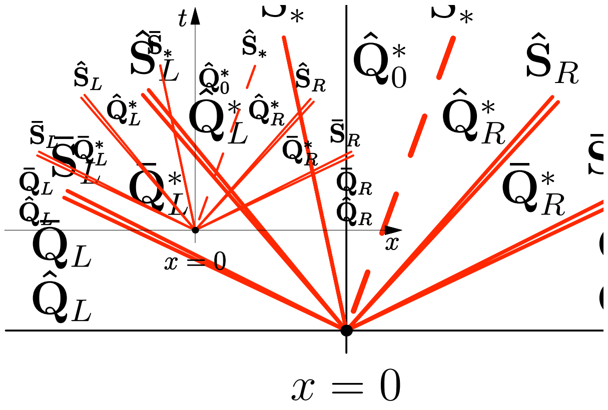
$$\hat{\rho}_{*L}(\hat{u}_{*L}) = \hat{\rho}_L\left(\frac{\hat{S}_L - \hat{u}_L}{\hat{S}_L - \hat{u}_{*L}}\right)$$

$$\hat{p}_{*L}(\hat{u}_{*L}) = \hat{p}_L + \hat{\rho}_L(\hat{S}_L - \hat{u}_L)(\hat{u}_{*L} - \hat{u}_L)$$

$$\hat{\rho}_{*R}(\hat{u}_{*R}) = \hat{\rho}_R\left(\frac{\hat{S}_R - \hat{u}_R}{\hat{S}_R - \hat{u}_{*R}}\right)$$

$$\hat{p}_{*R}(\hat{u}_{*R}) = \hat{p}_R + \hat{\rho}_R(\hat{S}_R - \hat{u}_R)(\hat{u}_{*R} - \hat{u}_R)$$

Thin-layer theory, case 1:   solid contact on the left



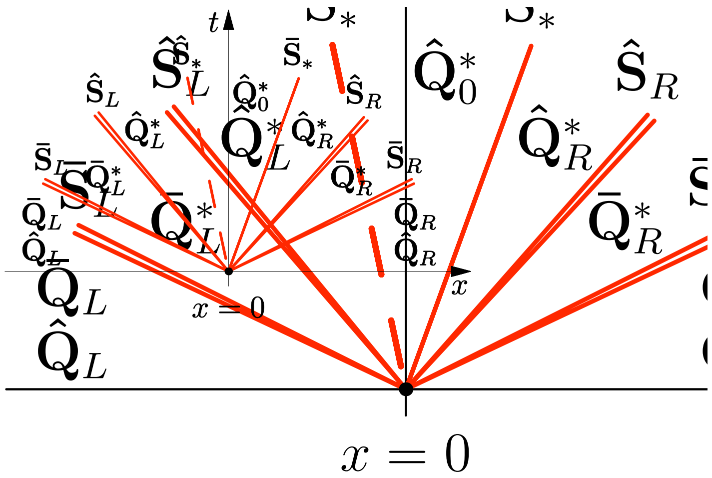
$$\bar{u}_{*L} - \bar{u}_{*R} = 0$$

$$\alpha_R \left(\frac{p_{*R}}{p_{*L}} \right)^{1/\gamma} (u_{*R} - \bar{u}_{*R}) - \alpha_L (u_{*L} - \bar{u}_{*L}) = 0$$

$$\bar{\alpha}_R \bar{p}_{*R} + \alpha_R p_{*R} - \bar{\alpha}_L \bar{p}_{*L} - \alpha_L p_{*L} + \alpha_L \rho_{*L} (u_{*L} - \bar{u}_{*L})(u_{*R} - \bar{u}_{*R}) = 0$$

$$\frac{\gamma p_{*R}}{(\gamma - 1)\rho_{*L}} \left(\frac{p_{*L}}{p_{*R}} \right)^{1/\gamma} + \tfrac{1}{2}(u_{*R} - \bar{u}_{*R})^2 - \frac{\gamma p_{*L}}{(\gamma - 1)\rho_{*L}} - \tfrac{1}{2}(u_{*L} - \bar{u}_{*L})^2 = 0$$

Thin-layer theory, case 2:   solid contact on the right



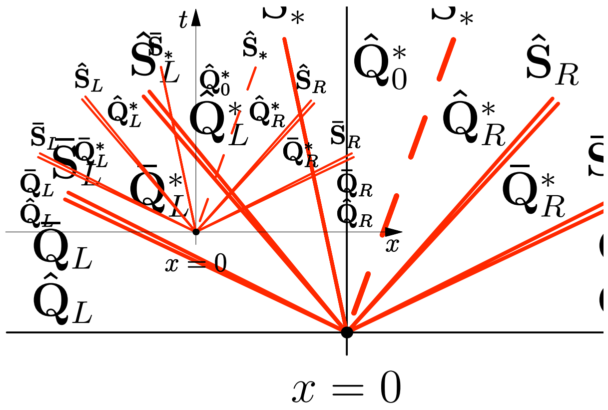
$$\bar{u}_{*R} - \bar{u}_{*L} = 0$$

$$-\alpha_L \left(\frac{p_{*L}}{p_{*R}} \right)^{1/\gamma} (u_{*L} - \bar{u}_{*L}) + \alpha_R (u_{*R} - \bar{u}_{*R}) = 0$$

$$\bar{\alpha}_R \bar{p}_{*R} + \alpha_R p_{*R} - \bar{\alpha}_L \bar{p}_{*L} - \alpha_L p_{*L} + \alpha_R \rho_{*R} (u_{*R} - \bar{u}_{*R})(u_{*R} - u_{*L}) = 0$$

$$-\frac{\gamma p_{*L}}{(\gamma - 1)\rho_{*R}} \left(\frac{p_{*R}}{p_{*L}} \right)^{1/\gamma} + \tfrac{1}{2}(u_{*R} - \bar{u}_{*R})^2 + \frac{\gamma p_{*R}}{(\gamma - 1)\rho_{*R}} - \tfrac{1}{2}(u_{*L} - \bar{u}_{*L})^2 = 0$$

Thin-layer theory, case 1: solid contact on the left



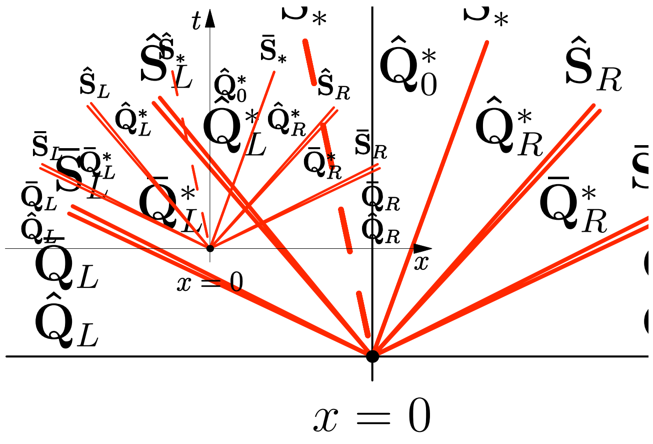
Non-linear system for 3 unknowns $\bar{u}, \bar{p}_{*L}, \bar{p}_{*R}$

$$\alpha_R \left(\frac{p_{*R}(u_{*R})}{p_{*L}(u_{*L})} \right)^{1/\gamma} (u_{*R} - \bar{u}) - \alpha_L (u_{*L} - \bar{u}) = 0$$

$$\bar{\alpha}_R \bar{p}_{*R}(\bar{u}) + \alpha_R p_{*R}(u_{*R}) - \bar{\alpha}_L \bar{p}_{*L}(\bar{u}) - \alpha_L p_{*L}(u_{*L}) + \alpha_L \rho_{*L}(u_{*L})(u_{*L} - \bar{u})(u_{*R} - u_{*L}) = 0$$

$$\frac{\gamma p_{*R}(u_{*R})}{(\gamma - 1)\rho_{*L}(u_{*L})} \left(\frac{p_{*L}(u_{*L})}{p_{*R}(u_{*R})} \right)^{1/\gamma} + \tfrac{1}{2}(u_{*R} - \bar{u})^2 - \frac{\gamma p_{*L}(u_{*L})}{(\gamma - 1)\rho_{*L}(u_{*L})} - \tfrac{1}{2}(u_{*L} - \bar{u})^2 = 0$$

Thin-layer theory, case 2:   solid contact on the right



Non-linear system for 3 unknowns $\bar{u}, \bar{p}_{*L}, \bar{p}_{*R}$

$$\alpha_L \left(\frac{p_{*L}(u_{*L})}{p_{*R}(u_{*R})} \right)^{1/\gamma} (u_{*L} - \bar{u}) - \alpha_R (u_{*R} - \bar{u}) = 0$$

$$\bar{\alpha}_R \bar{p}_{*R}(\bar{u}) + \alpha_R p_{*R}(u_{*R}) - \bar{\alpha}_L \bar{p}_{*L}(\bar{u}) - \alpha_L p_{*L}(u_{*L}) + \alpha_R \rho_{*R}(u_{*R})(u_{*R} - \bar{u})(u_{*R} - u_{*L}) = 0$$

$$\frac{\gamma p_{*L}(u_{*L})}{(\gamma - 1)\rho_{*R}(u_{*R})} \left(\frac{p_{*R}(u_{*R})}{p_{*L}(u_{*L})} \right)^{1/\gamma} - \tfrac{1}{2}(u_{*R} - \bar{u})^2 - \frac{\gamma p_{*R}(u_{*R})}{(\gamma - 1)\rho_{*R}(u_{*R})} + \tfrac{1}{2}(u_{*L} - \bar{u})^2 = 0$$

Non-linear algebraic system to solve.

It is enough to perform one iteration

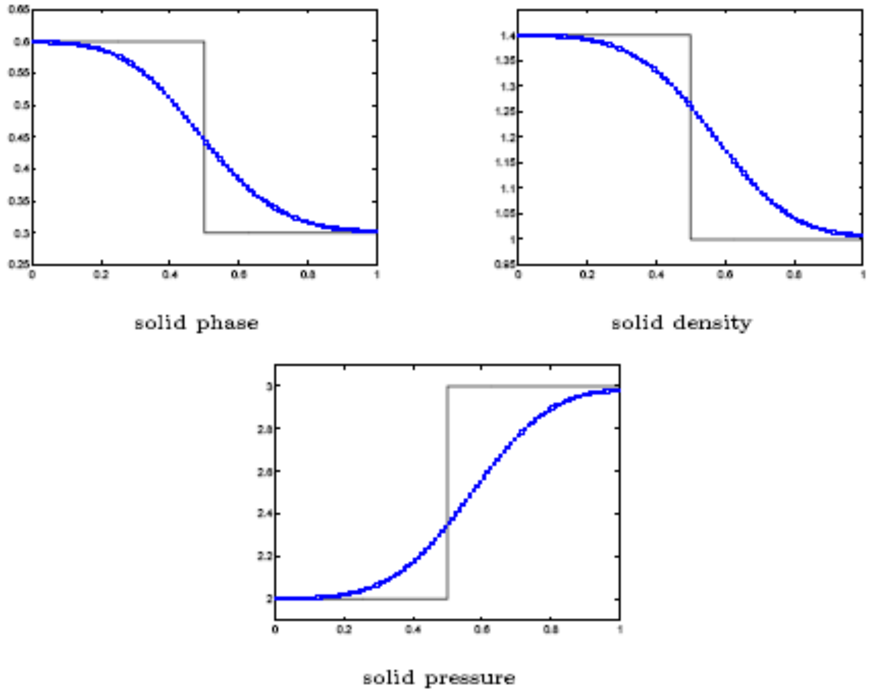
Predictor-corrector scheme

This HLLC-type solver is complete, also for the 3D case.
It accounts for all characteristic fields (11)

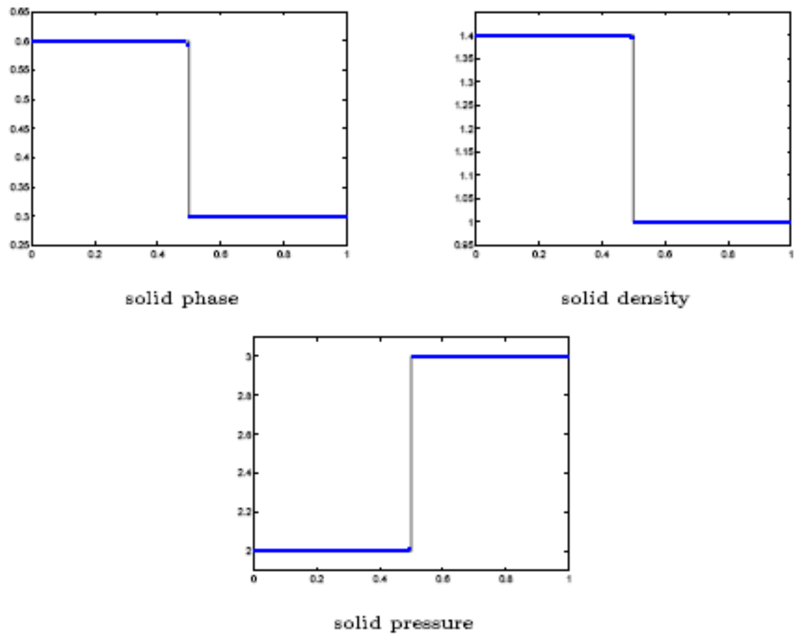
Solver used for 3 classes of schemes:
finite volumes,
DG finite elements and
a “new” version of path conservative

SA Tokareva and E F Toro. HLLC-type Riemann Solver for the Baer-Nunziato Equations of Compressible Two-Phase Flow. Journal of Computational Physics. (to appear, 2010)

A simple test problem: numerical results (FV method)



HLL



HLLC

Summary and concluding remarks

HLLC Riemann solver relies on a suitable wave model.
Here we have described the method as applied to:

3D Euler equations

2D shallo water equations

3D Baer-Nunziato equations of compressible two-phase flow

Further reading: chapter 10 of
Toro E F. **Riemann solvers and numerical methods for fluid dynamics**. Springer, Third Edition, 2010. Chapter 10+ REFERENCES THEREIN.