The Dynamics of Shallow Fluid Flows: Modeling and Numerical Analysis

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Part II of III:

Chapter 3 and Exercises

Yesterday: Part I

1. Inviscid Shallow Water Equations
2. High-Order Well-Balancing for Moving Equilibria

Today: Part II

3. Multi-Layer Systems

* **Exercises start**

Tomorrow: Part III

4. Conservative and Non-Conservative Aspects
5. Extended Shallow Water Models

* **Exercises continue**

Chapter 3

Multi-Layer Systems

Starting point:

- incompressible Euler equations
- hydrostatic pressure

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) + \partial_z(\rho w) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_z(\rho u w) &= -\frac{1}{F^2}\left(\rho(\eta)\,\partial_x \eta + \int_z^\eta \partial_x \rho\, d\zeta\right) \end{aligned}$$

incompressibility

$$\partial_x u + \partial_z w = 0$$

Modeling Assumption:

- $(K+1)$ interfaces η_k moving with the flow

$$\eta_0(x,t) = b(x)$$
$$\eta_K(x,t) = \eta(x,t)$$
$$w = \partial_t \eta_k + u \partial_x \eta_k \qquad \text{for } z = \eta_k$$

- horizontal layers Ω_k with constant density ρ_k

$$\Omega_k := \{(x,z,t) \,|\, \eta_{k-1}(x,t) < z < \eta_k(x,t)\}, \quad k = 1 \ldots K$$

$$\rho(x,z,t) \equiv \rho_k > \rho_{k+1} \quad \text{in} \quad \Omega_k$$

Definition: $f(x,z,t)$ integrable, $h_k := \eta_k - \eta_{k-1}$

$$\overline{f}_k(x,t) := \int_{\eta_{k-1}}^{\eta_k} f(x,z,t)dz \quad \textbf{depth – integral}$$

Transport Theorem: Let $f(x,z,t)$ be differentiable and satisfy kinematic boundary conditions at $z = \eta_{k-1}$ and $z = \eta_k$. Then

$$\int_{\eta_{k-1}}^{\eta_k} \left(\partial_t f + \partial_x(uf) + \partial_z(wf)\right) dz = \partial_t \overline{f}_k + \partial_x \overline{(uf)}_k$$

Proof:

$$\partial_t \overline{f}_k = (f\partial_t\eta)|_{\eta_{k-1}}^{\eta_k} + \int_{\eta_{k-1}}^{\eta_k} \partial_t f\, dz$$

$$\partial_x \overline{uf} = ((uf)\partial_x\eta)|_{\eta_{k-1}}^{\eta_k} + \int_{\eta_{k-1}}^{\eta_k} \partial_x(uf)\, dz$$

implies

$$\begin{aligned}
&\partial_t \overline{f}_k + \partial_x \overline{(uf)}_k \\
&= \int_{\eta_{k-1}}^{\eta_k} (\partial_t f + \partial_x(uf))\, dz + (f(\partial_t\eta + u\partial_x\eta))\,|_{\eta_{k-1}}^{\eta_k} \\
&= \int_{\eta_{k-1}}^{\eta_k} (\partial_t f + \partial_x(uf))\, dz + (fw)|_{\eta_{k-1}}^{\eta_k} \\
&= \int_{\eta_{k-1}}^{\eta_k} (\partial_t f + \partial_x(uf) + \partial_z(wf))\, dz
\end{aligned}$$

q.e.d.

Corollary:

$$\partial_t f + \partial_x(uf) + \partial_z(wf) = S$$

implies

$$\partial_t \overline{f}_k + \partial_x \overline{(uf)}_k = \overline{S}_k.$$

Example: $f = \rho$, so $S = 0$,

$$\partial_t \overline{\rho}_k + \partial_x \overline{q}_k = 0 \quad \text{variable density continuity equation}$$

where $q := \rho u$. Other forms, using $\rho \equiv \rho_k$:

$$\partial_t \overline{1}_k + \partial_x \overline{u}_k = 0$$

$$\partial_t h_k + \partial_x (h_k u_k) = 0$$

where

$$h_k := \overline{1}_k = \int_{\eta_{k-1}}^{\eta_k} 1 \, dz$$

is the height of the layer.

Example: $f = q$ (discharge), so

$$S = -\ \frac{1}{F^2}\partial_x p$$

and

$$\partial_t \overline{q}_k + \partial_x \overline{(uq)}_k = \ -\frac{1}{F^2}\overline{(\partial_x p)}_k$$

variable density momentum equation

Assumption: piecewise-constant vertical velocity-profile,

$$u(x,z,t) \equiv u_k(x,t) \qquad \text{for } (x,z,t) \in \Omega_k$$

Note:

$$\overline{(uq)}_k = \overline{q}_k^2/\overline{\rho}_k = \rho_k h_k u_k^2.$$

Preliminary summary:

Integrated mass and momentum:

for each layer Ω_k, $k = 1 \ldots K$

$$\partial_t \overline{\rho}_k + \partial_x \overline{q}_k = 0$$

$$\partial_t \overline{q}_k + \partial_x \left(\frac{\overline{q}_k^2}{\overline{\rho}_k} \right) = -\frac{1}{F^2} \overline{(\partial_x p)}_k$$

Kinematic boundary condition:

for each interface η_k, $k = 0 \ldots K$

$$w = \partial_t \eta_k + u \partial_x \eta_k$$

hydrostatic assumption

$$p(x,z,t) = p_a + \int_z^{\eta(x,t)} \rho\,(x,\zeta,t)d\zeta$$

interface pressure: for $k = 0 \ldots K$

$$p_k(x,t) \;:=\; p(x,\eta_k(x,t),t) \;=\; p_a + \sum_{l=k+1}^{K} \overline{\rho}_l(x,t)$$

pressure within layer: in Ω_k $(\eta_{k-1} < z < \eta_k)$,

$$p(x,z,t) = p_k(x,t) + \rho_k\,(\eta_k(x,t) - z)$$

pressure gradient within layer:

$$\partial_x p(x,z,t) = \partial_x p_k(x,t) + \rho_k\, \partial_x \eta_k(x,t)$$

layer-average of pressure gradient:

$$\begin{aligned}\overline{(\partial_x p)}_k &= h_k \partial_x p_k + \overline{\rho}_k\, \partial_x \eta_k \\ &= h_k \partial_x p_k + \overline{\rho}_k\, \partial_x (\eta_{k-1} + h_k) \\ &= \underbrace{h_k \partial_x p_k}_{\text{surface pressure}} + \underbrace{\partial_x \left(\frac{1}{2}\rho_k h_k^2\right)}_{\text{fluid pressure}} + \underbrace{\overline{\rho}_k \partial_x \eta_{k-1}}_{\text{bottom slope}}\end{aligned}$$

Single layer:

$$\overline{(\partial_x p)} = h \partial_x p_a + \partial_x \left(\frac{1}{2}\rho h^2\right) + \overline{\rho} \partial_x b$$

Summary: Integrated mass and momentum:

$$\partial_t \overline{\rho}_k + \partial_x \overline{q}_k = 0$$

$$\partial_t \overline{q}_k + \partial_x \left(u_k \overline{q}_k + \frac{1}{2F^2} h_k \overline{\rho}_k \right) = -\frac{1}{F^2} \left(h_k \partial_x p_k + \overline{\rho}_k \partial_x \eta_{k-1} \right)$$

where

$$p_k = p_a + \sum_{l=k+1}^{K} \overline{\rho}_l$$

$$\eta_{k-1} = b + \sum_{l=1}^{k-1} h_l$$

Example: **2-layer equations**: $\eta = \eta_2 > \eta_1 > \eta_0 = b$

$$\partial_t \overline{\rho}_k + \partial_x \overline{q}_k = 0 \qquad k = 1, 2$$

$$\partial_t \overline{q}_2 + \partial_x \left(u_2 \overline{q}_2 + \frac{1}{2F^2} h_2 \overline{\rho}_2 \right) = -\frac{1}{F^2} \left(h_2 \partial_x p_a + \overline{\rho}_2 \partial_x (b + h_1) \right)$$

$$\partial_t \overline{q}_1 + \partial_x \left(u_1 \overline{q}_1 + \frac{1}{2F^2} h_1 \overline{\rho}_1 \right) = -\frac{1}{F^2} \left(h_1 \partial_x (p_a + \overline{\rho}_2) + \overline{\rho}_1 \partial_x b \right)$$

Assume from now on that $\partial_x p_a = 0$

Properties:

- hyperbolic-elliptic 4x4 system
  - hyperbolic for small shears
  - elliptic for large shears

  $\longrightarrow$ rest of Chapter 3
- non-conservative

  $\longrightarrow$ Chapter 4
- interfaces treated automatically

  by conservative continuity equation

Limit of

- almost constant water level
- almost equal densities
- vanishing velocity difference

J. B. Schijf and J. C. Schonfeld (1953)

External and internal eigenvalues in dimensional variables:

$$\lambda_{ext}^{\pm} \approx \frac{u_1h_1 + u_2h_2}{h_1 + h_2} \pm (g(h_1 + h_2))^{\frac{1}{2}} \tag{1}$$

$$\lambda_{int}^{\pm} \approx \frac{u_1h_2 + u_2h_1}{h_1 + h_2} \pm \left[g' \frac{h_1h_2}{h_1 + h_2}\left(1 - \frac{(u_1 - u_2)^2}{g'(h_1 + h_2)}\right)\right]^{\frac{1}{2}}, \tag{2}$$

where

$$g' := (1 - r)g = \frac{\rho_1 - \rho_2}{\rho_1} g =: \varepsilon g$$

Remainder of this chapter

- derivation of approximate eigenvalues
- numerical examples

Exercise 1:

Rewrite the 2-layer equations as

$$\partial_t U + A\partial_x U = S$$

with

$$U := \begin{pmatrix} h_1 \\ u_1 \\ h_2 \\ u_2 \end{pmatrix} \qquad A := U := \begin{pmatrix} u_1 & h_1 & 0 & 0 \\ 1/F^2 & u_1 & \rho_2/(\rho_1 F^2) & 0 \\ 0 & 0 & u_2 & h_2 \\ 1/F^2 & 0 & 1/F^2 & u_2 \end{pmatrix}$$

Exercise 2:

Compute the characteristic polynomial

$$
\begin{aligned}
p(\lambda) = (1-\varepsilon)\alpha_1\alpha_2 K^2 \\
+ \left((\lambda - u_1)^2 - \alpha_1 K\right)\left((\lambda - u_2)^2 - \alpha_2 K\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_k &:= h_k / H \\
H &:= h_1 + h_2 \\
K &:= H / F^2
\end{aligned}
$$

Notation:

$$\bar{u} := (u_1 + u_2)/2$$

$$\tilde{u} := \alpha_1 u_1 + \alpha_2 u_2$$

$$\hat{u} := \alpha_2 u_1 + \alpha_1 u_2$$

$$v := u_2 - u_1$$

$$\varepsilon := (\rho_1 - \rho_2)/\rho_1$$

In the following exercises, always keep track of the orders

$$\mathcal{O}(v^m + \varepsilon^n)$$

which you neglected.

I did not do this on these slides, and the identies sometimes only hold approximately.

Exercise 3:

(i) Let $\kappa := \lambda - \tilde{u}$. Rewrite the characteristic polynomial als

$$q(\kappa) := p(\lambda(\kappa))$$

and expand it in orders of v and ε.

Verify that

Exercise 4:

Compute the approximate eigenvalues

$$\lambda_{ext}^{\pm} = \tilde{u} \pm K^{1/2}$$

Note:

2-layer system is hyperbolic $\iff$ $p(\lambda_0) > 0$

where λ_0 is the local maximum,

$$p'(\lambda_0) = 0$$
$$p''(\lambda_0) < 0$$

Draw a picture of p.

Exercise 5:

Verify that

$$p'(\lambda) = \lambda(\lambda - \bar{u})(\lambda - 2\bar{u})$$
$$- \frac{1}{2}\left[(\lambda - \hat{u})(K - 2u_1u_2) + vu_1u_2(\alpha_2 - \alpha_1)\right]$$

Therefore we are looking for the intersection of a cubic and a linear function.

Exercise 6:

Do one Newton step starting from $\lambda = \bar{u}$ and ending at $\tilde{\lambda}_0$,

$$\tilde{\lambda}_0 = \bar{u} - p'(\bar{u})/p''(\bar{u}).$$

Verify that

$$\tilde{\lambda}_0 = \bar{u} + \frac{\alpha_2 - \alpha_1}{4K} v^3$$

Exercise 7:

Verify that

$$p(\tilde{\lambda}_0) = \left(v^2\alpha_1\alpha_2 - \frac{K}{2}\right) - \frac{K^2}{4}(1 - 4\varepsilon\alpha_1\alpha_2)$$

Exercise 8:

Verify that $p(\tilde{\lambda}_0) > 0$ if

$$v^2 \leq \varepsilon K \tag{3}$$

so the system is hyperbolic. Verify that this coincides with the condition of Schijf and Schonfeld.

Exercise 9:

Find the internal eigenvalues as roots of the quadratic Taylor expansion of $p(\lambda)$ near $\tilde{\lambda}_0$.

Verify that

$$\lambda_{int}^{\pm} = \hat{u} \pm \left(\alpha_1 \alpha_2 \left(\varepsilon K - v^2 \right) \right)^{1/2}$$

Exercise 10:

Redo all this in your own, perhaps more straightforward way

:–) :–) :–)

Numerical experiments for bi-layer equations

- Strait of Gibraltar
- symmetric cross-section
- well-balanced scheme

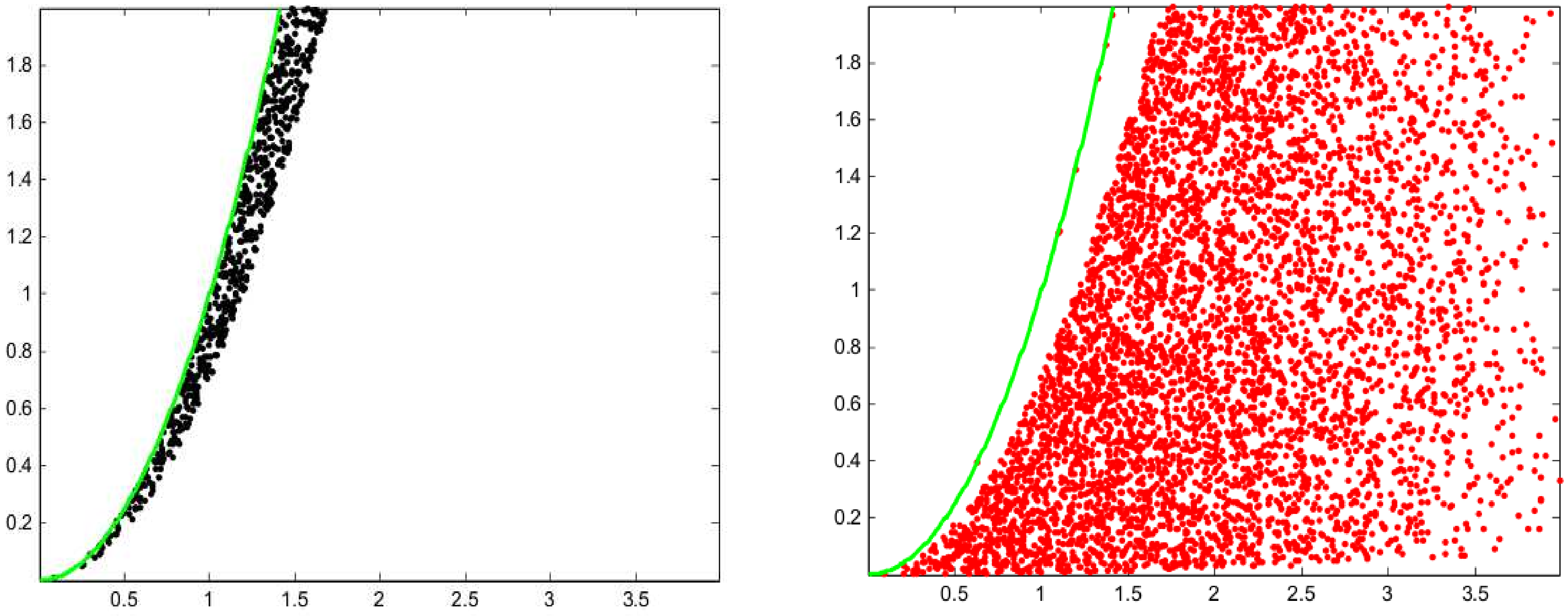
Castro, García-Rodríguez, Gonzalez-Vida, Macías, Parés, Vázquez-Cendón, JCP 2004

- p.25, Fig. 11: Geometry
- p.26, Fig. 13: Solution and eigenvalues

Numerical experiments showing hyperbolic regions

- Monte-Carlo sampling
- hyperbolic regions
- 3-layer hyperbolic regions

Computation by **Manuel Castro**



Introducing a third layer for non-hyperbolic two-layer states

Left: hyperbolic region Right: non-hyperbolic region