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High-order well-balanced finite volume schemes for 1d and 2d shallow water equations with Coriolis forces

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In this work we present two novel well-balanced high-order finite volume schemes for the shallow-water system with Coriolis source term.

We consider first the 1D shallow-water system with Coriolis term that could be written as a hyperbolic balance law as follows:

$$\partial_t \mathbf{q} + \partial_x \mathbf{f}(\mathbf{q}) = \mathbf{s}(\mathbf{q}), \tag{1}$$

where $\mathbf{q} = (h, hu, hv)$, $\mathbf{f}(\mathbf{q}) = (hu, hu^2 + gh^2/2, huv)$ and $\mathbf{s}(\mathbf{q}) = (0, fhv - gh\partial_x z, -fhu)$. As usual, h denotes the water depth, u and v are the two components of the averaged horizontal velocity, z designates the topography and is supposed to be a given function, g is the constant gravitational acceleration and f the Coriolis parameter.

Now, the main objective is to define a high-order (≥ 3) exactly well-balanced finite volume scheme for the stationary solutions of (1). By exactly well-balanced we mean a method that is able to preserve the cell-averages, computed by a quadrature formula, of the exact stationary solutions of (1). Here we follow the general procedure described in [5], where the well-balanced property is achieved by combining a standard consistent first order numerical flux for the homogeneous system, with a exactly well-balanced reconstruction operator. In particular, the method proposed here extends the one presented in [7], where only first and second order schemes are proposed.

Next, we consider the 2D shallow-water system with a Coriolis source term that could be written as follows:

$$\partial_t \mathbf{q} + \partial_x \mathbf{f}(\mathbf{q}) + \partial_y \mathbf{g}(\mathbf{q}) = \hat{\mathbf{s}}(\mathbf{q}), \tag{2}$$

where $\mathbf{g}(\mathbf{q}) = (hu, huv, hv^2 + gh^2/2)$ and $\hat{\mathbf{s}}(\mathbf{q}) = (0, -gh\partial_x z(x, y) + fhv, -gh\partial_y z(x, y) - fhu)$.

The design of general high-order exactly well-balanced finite volume schemes for 2D balance laws is harder than in 1D, as stationary solutions are now solutions of 2D nonlinear PDEs. In the literature, we can find different strategies to design well-balanced schemes for multi-dimensional problems, mainly in the framework of shallow flows, see for instance [1, 2, 3, 4, 8, 9, 10].

In the framework of 2D shallow-water flows with Coriolis forces we are interested in the design of first and second order finite volume schemes that are able to accurately approximate stationary solutions corresponding to geostrophic equilibrium, that is, solutions that satisfy (see for example [6])

$$\nabla \cdot \boldsymbol{u} = 0,\ \partial_x \eta = \frac{f}{g} v,\ \partial_y \eta = -\frac{f}{g} u, \tag{3}$$

where $\eta = h + z$ is the free surface.

Observe that those equilibria must satisfy the free divergence equation, which requires to solve an elliptic problem to determine the velocity field. Instead of solving a global problem, our approach is to consider a finite volume scheme that preserves a discrete version of $\nabla \cdot \boldsymbol{u} = 0$, that may be imposed locally. In particular, in 2D rectangular grids, we approximate

$$\nabla \cdot \boldsymbol{u}(x_i, y_j) \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}, \tag{4}$$

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where we identify cell-averages with point values at the centre of the cells.

We consider now that, locally, at each cell, the velocity vector field satisfies: it is a piece-wise linear function, verifies (4) and its point values are the *closest* to the ones available in the *stencil*. This problem has a unique solution in the sense of the least squares. Next, the free surface is recovered by integrating the second and third equations of system (3).

The resulting numerical scheme is exactly well-balanced if the velocity vector field is linear and, otherwise, it is well-balanced.

In figure 1 we show the comparison between the solutions obtained with the here proposed scheme, and a standard numerical method that is well-balanced for water at rest, but not for the geostrophic equilibrium. Observe how our first and second order schemes are more accurate than the standard first and second order ones. This plot is a 1D cut at $y = 0$ for the water depth h, at time $T = 10s$, for a domain $[-1,1] \times [-1,1]$ discretized with a 200×200 mesh.

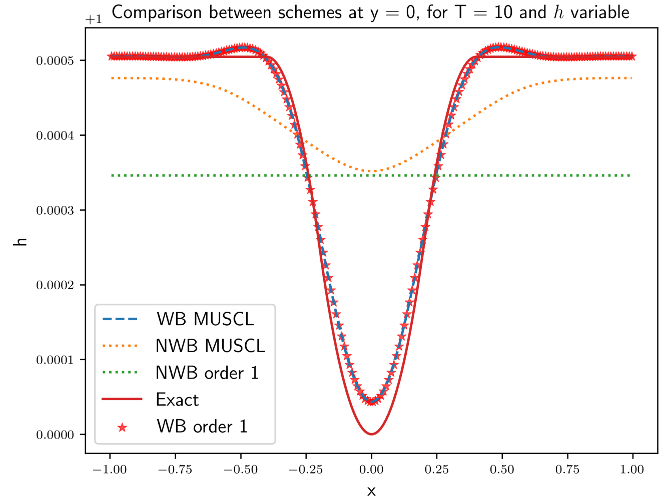

Figure 1: Comparison between well-balanced and non well-balanced solutions with the exact geostrophic vortex for reconstructions of order 1 and 2.

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