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Commutative properties for conservative space-time DG discretizations of optimal control problems involving regularized hyperbolic problems

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We consider optimal control problems with distributed control on a space-time domain $Q = \Omega \times (0,T)$ with $\Omega = (x_L, x_R) \subset \mathbb{R}$ and the state equation being given by a regularized hyperbolic equation. We first focus on the case of the viscous Burgers equation, i.e., we consider

$$\min J(q,u) = \frac{1}{2}\|u - u_d\|^2_{L^2(Q)} + \frac{\alpha}{2}\|q\|^2_{L^2(Q)}$$

$$\text{subject to} \quad \begin{aligned} u_t + \left(\tfrac{1}{2}u^2\right)_x - \varepsilon u_{xx} &= q + g && \text{in } Q, \\ u(x_L,\cdot) = u(x_R,\cdot) &= 0 && \text{in } (0,T), \\ u(\cdot,0) &= u^0 && \text{in } \Omega. \end{aligned}$$

Here, q denotes the control, u the state, and g a source term. The parameter α enforces the Tikhonov regularization. We assume $\varepsilon > 0$ to be constant. We are interested in the numerical behavior for small values of ε.

Generally, there are two approaches for discretizing an optimal control problem:

- *optimize-then-discretize* (OD): one first sets up the optimality system in a continuous setting and then discretizes the resulting equations;
- *discretize-then-optimize* (DO): one first discretizes the optimal control problem including the state equation and then sets up the optimality system in a discrete setting.

Depending on the discretization, the OD approach has a consistent discretization of the adjoint equation but might result in inexact discrete gradients, whereas the DO approach yields exact discrete gradients but might have an inconsistent discretization of the adjoint. In an ideal case, both approaches commute and one obtains an exact discrete gradient and a consistent discretization of the adjoint.

In this work, we present a spacetime discontinuous Galerkin (DG) discretization for the above optimal control problem, which has this property [1]. Both the state and adjoint equation are discretized using an upwind flux in time and the symmetric interior penalty discretization (SIPG) for the diffusion term. Our focus is on the discretization of the convection term. We discretize both the convection term in the state and in the adjoint equation in a conservative way and show that one can restore commutativity by adding a suitable source term. We follow the ideas of well-balanced schemes for deriving this source term. Numerical results demonstrate better results when including this source term and thereby restoring commutativity compared to not doing that.

We then proceed to the case of the state equation being given by the compressible Navier-Stokes equations. We use entropy variables as degrees of freedom and therefore consider the state equation

$$\mathbf{U}(\mathbf{V})_t + \mathbf{F}(\mathbf{V})_x - (\mathbf{A}(\mathbf{V})\mathbf{V}_x)_x = \mathbf{q} + \mathbf{g},$$

with $\mathbf{U}$ being the conserved variables and $\mathbf{V}$ being the entropy variables. As a result, the diffusion matrix $\mathbf{A}$ is symmetric, which

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is beneficial for commutativity. We discretize the state equation following the work in [2]. We will discuss the current status quo concerning the well-posedness of the adjoint system and the commutativity property of DO and OD.

References

[1] X. Kerkhoff, S. May. Commutative properties for conservative space-time DG discretizations of optimal control problems involving the viscous Burgers equation. *Math. Control Relat. Fields*, Online First, 2021.

[2] S. May. Spacetime discontinuous Galerkin methods for solving convection-diffusion systems. *ESAIM Math. Model. Numer. Anal.*, 51(5): 1755–1781, 2017.