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# The Riemann Problem for a Weakly Hyperbolic Two-Phase Flow Model of a Dispersed Phase in a Carrier Fluid

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Multi-phase flows occur in many natural and engineering applications. Consider the following phases of matter: gas, liquid and solid. We are interested in flows in which one phase is dispersed in a carrier phase. With these phases, the two-phase flows can be separated into three categories: gas-liquid, gas-solid and liquid-solid flows. The first category includes so called "bubbly flows" of gas bubbles in a liquid carrier. One finds these, for example, in chemical reactors. Ship manufacturers are interested to know the effects of bubbles formed in water due to cavitation because they cause substantial damage to propellers. In this category, a second flow form is gas-droplet flows typically found in atmospheric physics, where cloud formation is of considerable interest. It involves liquid water and its vapor, as well as a mixture of other gases. Other examples are sprays with various applications. The second category is gas-solid particle flows. These occur in astrophysics during star formation or in powder applications in the industry. The third category consists of solid-liquid flows, which play an essential role in transport processes like sedimentation. Here, the solid phase should only occur in the role of the dispersed phase. Otherwise, if the solid takes the role of the carrier phase, one has the field of porous media flows, which is quite different in nature. In the beverage industry, coffee beans are sugar-coated using a fluidized bed granulator. This process involves a mixture of solid particles, liquid sprays and gas flow. All such processes are modeled using two-phase or multi-phase flow models.

We study the two-phase flow model proposed by Dreyer, Hantke and Warnecke [1]. We are not particularly interested in the evolution of single bubbles or droplets. Therefore, we would like to describe the evolution of macroscopic quantities. Starting from basic physical laws, there are many techniques to arrive at a macroscopic model. On a macroscopic level, often mixture models are applied. In these models, both phases may be present at any point in space and time. This is taken into account by a volume fraction density functions. The equations are usually derived from microscopic considerations using averaging or homogenization techniques. The resulting model describes the evolution of a mixture of a dispersed phase of small ball-shaped bubbles or droplets, immersed in a carrier fluid and it is completely in divergence form.
For the mathematical analysis here, we will only consider one space dimension, neglect the phase exchange terms handling phase transitions and assume isothermal flow. In this form, it is a *weakly hyperbolic* system of conservative partial differential equations. By this, we mean that all eigenvalues are real but there is at least one multiple eigenvalue that does not have a full set of eigenvectors. Existence results for hyperbolic systems of conservation laws in one space dimension require strict hyperbolicity, i.e. a full set of distinct real eigenvalues. Therefore, the existing theory does not apply to our system.
Due to the lack of a general theory, we will consider Riemann problems only and not general initial data. We will analyze the elementary wave structure and use Riemann invariants as well as Rankine-Hugoniot jump conditions to determine a highly nonlinear system of algebraic equations connecting the initial states to each other. We then have to find a solution to these nonlinear systems to provide a solution and use monotonicity arguments to show uniqueness.
Our work includes the first analysis of the Riemann problem of the two-phase flow model considered. We perform the eigenstate analysis on the dispersed phase alone as well as the full two-phase system of equation. The wave types and all possible wave patterns are found. The solutions include classical waves such as rarefactions, shock waves and contact discontinuities. Interestingly, in some cases, it is possible to obtain solutions involving vaporless states as well as the non-classical $\delta$-shock waves.
This work exclusively addresses the Riemann problem. Our aim is to understand the mathematical structure of the conservation part of the model. Solutions to the Riemann problem are found by solving the highly nonlinear systems of algebraic equations

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mentioned above. These solutions are self-similar and uniquely determined by the initial data. All solutions are given implicitly and uniqueness was shown using monotonicity arguments. The final result is a set of inequalities for the relative velocity between the two phases involved. To ensure the uniqueness of the solution, this relative velocity should be a certain amount smaller than the sound speed in the carrier phase. Its explicit value depends on the chosen equation of state and the parameters therein, as well as the initial data used. These bounds on the velocity are not sharp but give a sufficient criterion to ensure the uniqueness of the solution.

We study bubbles in a liquid carrier as well as droplets or dust particles in a vapor carrier. In a gas, the equation of state (EOS) for isothermal flow yields the pressure as a linear function of the density. For a liquid, the simplest realistic assumptions lead to an affine function for the EOS. The analysis for an affine linear equation of state is much more complicated. This is a key point of this work. Nonetheless, all possible wave configurations are discussed, the implicit functions to find a solution are given and the inequalities assuring monotonicity are stated as well.

Therefore, the work takes a first step from a linear equation of state towards more general equations of state, which is very important with regard to applications. Commonly used equations of state like the Tait equation or the stiffened gas equation are included in our analysis. It is remarkable how the slight change in the equation of state complicated the analysis considerably. Initial data and explicit solutions are given for all relevant cases. We choose in particular physically reasonable values.

The theoretical results which we will describe are already published in [2, 3].

Considering the numerical part of this work, we obtained simulations for the cases considered in the analytical sections of this work. We used a second-order MUSCL-Hancock scheme with the MINBEE limiter and the HLL approximate Riemann solver.

Since we only have an analytical solution in the one dimensional case, we left out simulations of higher space dimensions, even though we used classical split methods to obtain them. Only in one space dimension, we can directly compare analytical and numerical results. The numerical simulations could be understood as a confirmation of the analytical calculations done. On the other hand, the analytical solution is a tool to verify numerical schemes in one space dimension and then generalize them to higher space dimensions in the hope that they will still approximate the exact solution.

We also tried to improve the classical HLL approximate Riemann solver by introducing the GHLL solver in [4]. This solver improved the resolution of $\delta$-shocks and contacts in the dispersed phase using the information from the explicit analytical solution in this phase. However, it led to oscillations in the carrier phase quantities. The construction of an improved Riemann solver for the model considered is therefore still an open problem.

We would also like to use solvers which use the eigenstructure of the quasi-linear system of conservation laws like the Roe scheme. But since the system of conservation laws under consideration is only weakly hyperbolic, we lack an eigenvector, which does not allow the usage of such numerical schemes. At the moment, we have two different ideas of how to overcome this difficulty. To fully treat the numerical investigation of this model, we want to write a follow-up paper on this issue in the near future.

## References


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