XVIII INTERNATIONAL CONFERENCE ON HYPERBOLIC PROBLEMS
THEORY, NUMERICS, APPLICATIONS
Málaga, June 20-24, 2022

A local a-posteriori error estimator for systems of hyperbolic conservation laws

Jan Giesselmann [*], Aleksey Sikstel [†]

We seek approximations of solutions to a one-dimensional system of strictly hyperbolic conservation laws defined on a space-time cylinder $\Omega_T := [0,T] \times \Omega$ with $\Omega \subset \mathbb{R}$,

$$\begin{aligned} \boldsymbol{u}_t(t,x) + \boldsymbol{f}(\boldsymbol{u}(t,x))_x = 0, \quad (t,x) \in \Omega_T \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x). \end{aligned} \tag{1}$$

Here $\boldsymbol{u} : \Omega_T \to \mathbb{D} \subset \mathbb{R}^m$ denotes the conserved quantity, $\boldsymbol{f} \in C^2(\mathbb{D}, \mathbb{R}^m)$ the flux function and $\boldsymbol{u}_0 \in L^\infty(\Omega, \mathbb{D})$ is the initial condition.

In practice, solutions of systems of hyperbolic conservation laws turn out to be of heterogeneous nature: in some regions solutions are smooth whereas in other regions discontinuities may occur. Thus, numerical simulations of these systems profit from adaptive mesh refinement avoiding waste of time and resources. To obtain a dynamic grid adaptation strategy we focus on rigorous a-posteriori error estimators. In general, this is a difficult task since there are no stable variational formulations for systems of hyperbolic PDEs yet developed. The recent, seminal work by Bressan and coauthors, [1], presents a novel estimator for one-dimensional systems of hyperbolic conservation laws on space-time slabs. In this work a local bound on the error is presented that is zero whenever the solution is exact.

We consider first-order finite volume schemes. Let the space-time domain Ω_T be discretized by grid points (t^n, x_j) where $j \in \{1, \dots, J\}$ and $n \in \{1, \dots, N\}$. We denote the corresponding time-steps by $\Delta t^{n+\frac{1}{2}} := t^{n+1} - t^n$ and cell widths by $\Delta x_{j+\frac{1}{2}} := x_{j+1} - x_j$. Finite volume schemes approximate the solution in each cell by spatial averages

$$\widehat{\boldsymbol{u}}^n_{j+\frac{1}{2}} := \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \boldsymbol{u}(t^n, x)\, dx,$$

that evolve in time on each time level t^n where $\widehat{\boldsymbol{u}}^0$ consists of piece-wise averages of the initial condition $\boldsymbol{u}_0$. The conservative form of the time-evolution of these spatial averages is given by

$$\widehat{\boldsymbol{u}}^{n+1}_{j+\frac{1}{2}} = \widehat{\boldsymbol{u}}^n_{j+\frac{1}{2}} - \frac{\Delta t}{\Delta x} \left(\widehat{\boldsymbol{f}}^n_{j+1} - \widehat{\boldsymbol{f}}^n_j \right), \tag{2}$$

where $\widehat{\boldsymbol{f}}^n_j := \widehat{\boldsymbol{f}}(\widehat{\boldsymbol{u}}_{j-1}, \widehat{\boldsymbol{u}}_j) : \mathbb{D} \times \mathbb{D} \to \mathbb{R}^m$ is the numerical flux that approximates $\boldsymbol{f}$ at the discontinuity x_j. Note that, for the sake of brevity, we are going to impose periodic boundary conditions in the following, i.e. $\boldsymbol{u}(t, x_0) = \boldsymbol{u}(t, x_N)$ for all $t \in [0, T]$.

We divide the space-time cylindrical domain Ω_T into rectangular space-time cells $\Omega_T = \bigcup_{j,n} K^n_j$. Each cell has vertices $\nu^m_l := (t^m, x_l)$ with $m \in \{n, n+1\}$ and $l \in \{j, j+1\}$. As in [1] we define the numerical solution on the whole domain Ω_T. We assume that the finite volume solution in the interior of K^n_j is constantly extended with the value of the earlier time level, i.e. for $(t,x) \in (t^n, t^{n+1}) \times (x_j, x_{j+1})$ we have $\widehat{\boldsymbol{u}}(t,x) = \widehat{\boldsymbol{u}}^n_{j+\frac{1}{2}}$.

[*]Dept. of Mathematics, TU Darmstadt, Dolivostraße 15, 64293 Darmstadt, Germany. Email: giesselmann@mathematik.tu-darmstadt.de
[†]Dept. of Mathematics, TU Darmstadt, Dolivostrße 15, 64293 Darmstadt, Germany. Email: sikstel@mathematik.tu-darmstadt.de

Let the set of edges of K_j^n be denoted by $\mathbb{E}\left(K_j^n\right)$, where

$$
\begin{aligned}
&\mathbb{E}\left(K_j^n\right) := \{S_{j,\pm}, S^{n,\pm}\},\\
&S_{j,+} := \overline{\nu_j^n \nu_j^{n+1}}, S_{j+1,-} := \overline{\nu_{j+1}^n \nu_{j+1}^{n+1}},\\
&S^{n,+} := \overline{\nu_j^n \nu_{j+1}^n}, S^{n+1,-} := \overline{\nu_j^{n+1} \nu_{j+1}^{n+1}},
\end{aligned} \tag{3}
$$

and the signs indicate negative or positive limits in the normal direction, since the values of the approximate solutions might jump at a cell interface. The limits in (3) are set to the interior side of the space-time cell K_j^n.

In [1] estimates of the L_1-error for first-order finite volume schemes for Cauchy-problems are presented provided that, among other conditions, the weak residuum of the numerical solution can be bounded by the total variation of the solution and other quantities, cf. condition $\mathbf{P}_\varepsilon$ therein. These estimates are based, among others, on the weak residua on slices $[\tau, \tau'] \times \mathbb{R}$ that cannot be split into single cells directly. The idea presented here is to localize the weak residuum and to find its inexpensively computable formulation that can be used as the basis for a local error estimator of a first-order finite volume scheme. In addition, the condition $\mathbf{P}_\varepsilon$ enters various estimates in the analysis of the L^1-error presented in [1] that can be applied onto the localized version.

The construction of a local version of the weak residuum on a space-time cell K_j^n requires fluxes at the edges $S_{j,+}$ and $S_{j+1,-}$. The most natural choice is to assign the corresponding numerical fluxes there. Thus, the local weak residuum operator on K_j^n is defined as

$$
\begin{aligned}
&\mathcal{B}_j^n : L^1(K_j^n) \times W^{1,\infty}(K_j^n) \to \mathbb{R}^m,\\
&\mathcal{B}_j^n[\widehat{\boldsymbol{u}}, \varphi] := \int_{x_j}^{x_{j+1}} \widehat{\boldsymbol{u}}_{j+\frac{1}{2}}^n \varphi(t^n, x)\, dx - \int_{x_j}^{x_{j+1}} \widehat{\boldsymbol{u}}_{j+\frac{1}{2}}^{n+1} \varphi(t^{n+1}, x)\, dx\\
&\qquad + \int_{K_j^n} \widehat{\boldsymbol{u}}_{j+\frac{1}{2}}^n \varphi_t(t,x) + \boldsymbol{f}(\widehat{\boldsymbol{u}}_{j+\frac{1}{2}}^n) \varphi_x(t,x)\, dx\, dt\\
&\qquad + \int_{t^n}^{t^{n+1}} \widehat{\boldsymbol{f}}_j^n \varphi(t, x_j)\, dt - \int_{t^n}^{t^{n+1}} \widehat{\boldsymbol{f}}_{j+1}^n \varphi(t, x_{j+1})\, dt
\end{aligned} \tag{4}
$$

where $\varphi \in W^{1,\infty}(K_j^n)$ are test functions and $\widehat{\boldsymbol{f}}$ numerical fluxes. Note that for a fixed finite volume solution $\widehat{\boldsymbol{u}}$ the operator $\mathcal{B}_j^n[\widehat{\boldsymbol{u}}, \cdot\,]$ is linear. The local a-posteriori error estimator is then recovered as an expression of the local weak residuum

$$
\eta_j^n(\widehat{\boldsymbol{u}}) := \sup_{\|\psi\|_{W^{1,\infty}\left(K_j^n\right)}=1} \left|\mathcal{B}_j^n[\widehat{\boldsymbol{u}}, \psi]\right|. \tag{5}
$$

together with a measure for the oscillation of the solution in the cell. In order to compute η we construct a specific set of test functions ψ that preserve the suprema of $\mathcal{B}$ on $W^{1,\infty}$. Next, we prove that the local L_1-error is bounded by η for first-order finite volume approximate solutions and, in addition, investigate the decay rates of the estimator. Finally, we propose an algorithm for evaluation of η and employ the equal distribution strategy, see [2], to construct an adaptive scheme.

References

[1] Bressan, A., Chiri, M. T., Shen, W. A posteriori error estimates for numerical solutions to hyperbolic conservation laws. *Archive for Rational Mechanics and Analysis*, 2021, 241(1), 357â402.

[2] Ohlberger, M. A review of a posteriori error control and adaptivity for approximations of non-linear conservation laws. *International journal for numerical methods in fluids*, 2009, 59(3), 333â354.