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Hyperbolic Techniques in Epidemiological Modeling

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The recent pandemic lead to the development of a variety of new epidemiological models. The search for optimal lockdown policies or vaccination strategies requires, at least, that stability estimates for the solutions to these models be available. However, in several instances, also well posedness appeared not yet covered in the literature.

The standard starting point in this area of virus spreading modeling is the so called SIR model, that dates back to the classical work [6]:

(1) $$\boxed{S} \to \boxed{I} \to \boxed{R} \qquad \left\{ \begin{array}{rcl} \dot S &=& -\rho\, I\, S \\ \dot I &=& \rho\, I\, S - (\vartheta + \mu) I \\ \dot R &=& \vartheta\, I \end{array} \right.$$

where, as usual, S, I and R are the Susceptible, Infected and Recovered individuals; ρ describes the spreading of the disease, ϑ the recovery rate and μ the mortality rate of infected individuals. The independent variable is, as usual, time. First, we discuss the extension of (1) presented in [4], namely

(2) $$\left\{ \begin{array}{l} \partial_t S + \partial_a S + \mathrm{div}_x(v_S\, S) + \mu_S\, S = - \int_{\mathbf{R}_+} \int_{\mathcal{X}} \rho(t,a,\alpha,x,\xi)\, I(t,\alpha,\xi)\, d\xi\, d\alpha\; S(t,a,x) \\ \partial_t I + \partial_a I + \mathrm{div}_x(v_I\, I) + \mu_I\, I = \int_{\mathbf{R}_+} \int_{\mathcal{X}} \rho(t,a,\alpha,x,\xi)\, I(t,\alpha,\xi)\, d\xi\, d\alpha\; S(t,a,x) - \kappa\, I - \vartheta\, I \\ \partial_t H + \partial_a H + \mu_H\, H = + \kappa\, I - \eta\, H \\ \partial_t R + \partial_a R + \mathrm{div}_x(v_R\, R) + \mu_R\, R = + \vartheta\, I + \eta\, H\,. \end{array} \right.$$

Here H stands for those infected individuals that are isolated, possibly Hospitalized, and do not propagate the infection. The independent variables are, besides time, x for the spatial position and a for age. The other parameters and functions above are typically age, time and space dependent and describe, as usual, mortality rates ($\mu_s, \mu_I, \mu_H, \mu_R$), movements ($v_S, v_I, v_R$), infection propagation among different age classes (ρ), lockdown rate (κ), recovery rates (ϑ, η). An obvious restriction on these terms is, for instance, that

$$\|x - \xi\| > r \implies \rho(t,a,\alpha,x,\xi) = 0\,,$$

for a suitable positive distance r. Note that in the equation for H, clearly, no spatial movement is allowed. Lockdown policies can be simulated and compared through numerical ibntegrations that will be discussed during the talk.

Then, on the basis of [3], we tackle the description of vaccination strategies and of their effects within the more general framework of compartmental models with intra–compartmental dynamics:

(3) $$\begin{array}{ccccc} & & \boxed{V(0) \to V(T_*)} & & \\ & \nearrow & \downarrow & \searrow & \\ \boxed{S} & \to & \boxed{I} & \to & \boxed{R} \\ & & \downarrow & & \end{array} \qquad \left\{ \begin{array}{r} \dot S = -\rho_S\, I\, S - p(t,S) \\ \partial_t V + \partial_\tau V = -\rho_V\, I\, V \\ \dot I = (\rho_S\, S + \int_0^{T_*} \rho_V\, V) I - \vartheta\, I - \mu\, I \\ \dot R = \vartheta\, I + V(t,T_*) \\ V(t,0) = p\,(t, S(t))\ . \end{array} \right.$$

Above, we use essentially the same notation as in (2). Note the introduction of the variable τ ranging in $[0, T_*]$ and describing the time since vaccination, with T_* breng the time necessary for the vaccine to provide full immunization. The compartmental

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scheme on the left above shows that the evolution of the immunization effect is an example of a dynamics within the compartment of the V variable. The extension of (3) to the case of a vaccination campaign consisting of different concurrent vaccines allows to compare different vaccination strategies.

More generally, the introduction of a dynamics within compartments can, for instance, smoothen the evolution between different stages of a complex model. An example provided in [3] concerns the SIDARTHE model [5].

Finally, we deal with the wellposedness and stability estimates for a wide class of equations of interest in epidemiology, such as

$$\begin{cases} \partial_t u^h + \operatorname{div}_x \left(v^h(t,x)\, u^h \right) = g^h\left(t, x, u(t,x), u(t)\right) & (t,x) \in \mathbf{R}_+ \times \mathcal{X} \\ u^h(t,\xi) = u^h_b\left(t, \xi, u(t)\right) & (t,\xi) \in \mathbf{R}_+ \times \partial\mathcal{X} \\ u^h(0,x) = u^h_o(x) & x \in \mathcal{X}\,, \end{cases} \tag{4}$$

where $h = 1, \ldots, k$, see [2]. The above initial boundary value problem contains nonlocal terms (above, the dependence on $u(t)$ is assumed to of a nonlocal, i.e. functional, nature) in the right hand side as well as in the boundary conditions. We provide detailed wellposedness and stability estimates that apply to various instances of (4) considered in the literature, in the framework of epidemiological models such as (1), as well as in cell division models [1, 9] or age and phenotypically structured population models [7, 8].

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