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Fine properties of weak solutions of Burgers equation and applications to a singularly perturbed variational problem

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We consider bounded weak solutions to the inviscid Burgers equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0.$$

A pioneering result in the theory of conservation laws establishes the well-posedness of the associated Cauchy problem in the class of bounded entropy solutions, namely bounded functions satisfying

$$\mu_\eta := \partial_t \eta(u) + \partial_x q(u) \leq 0 \qquad \text{in } \mathcal{D}' \tag{1}$$

for every convex entropy $\eta : \mathbb{R} \to \mathbb{R}$ and corresponding flux $q : \mathbb{R} \to \mathbb{R}$ defined up to constants by $q'(v) = \eta'(v)v$. We are interested in the more general class of weak solutions with finite entropy production, where the distribution μ_η in (1) is only required to be a locally finite Radon measure (without constraints on its sign). Although weak solutions with finite entropy production are not locally of bounded variation, they share with BV functions most of their fine properties: in particular a one dimensional rectifiable set J of shocks is identified in [DLOW03, Lec05]. The main result of this talk is that the measure μ_η is concentrated on this shock set J. The analysis builds on a recent tool, called *Lagrangian representation*, recently introduced in this setting in [Mar22], which consists in an extension of the method of characteristics to the non smooth setting. By means of this tool, we can decompose the entropy production measures μ_η along characteristics and then deduce precise information on their structure.

The motivation of this result is the study of the asymptotic behaviour as $\varepsilon \to 0^+$ of the following functionals:

$$F_\varepsilon(u, \Omega) := \int_\Omega \left(\varepsilon |\nabla^2 u| + \frac{1}{\varepsilon} \left|1 - |\nabla u|^2\right|^2\right) dx, \qquad \text{where } \Omega \subset \mathbb{R}^2. \tag{2}$$

Limits of functions u_ε with uniformly bounded energy solve the eikonal equation $|\nabla u| = 1$, which can be interpreted as an $\mathbf{S}^1$-valued conservation law. In particular the notion of entropy can be introduced also in this setting and it turns out that the candidate Γ-limit F_0 can be described in terms of the entropy dissipation measures. The notion of Lagrangian representation, as well as several tools from the theory of conservation laws, can be adapted to this setting and several results on the structure of weak solutions with finite entropy production of Burgers equation can be transferred to the solutions of the eikonal equation relevant in the study of (2). Although a full rectifiability result as in the case of Burgers equation is still not available, we can prove some partial results in this direction [Mar21]. As a consequence we obtain that if Ω is an ellipse and u_ε are minimizers of $F_\varepsilon(\cdot, \Omega)$ under appropriate boundary conditions, then

$$u_\varepsilon \to \bar{u} := \mathrm{dist}(\cdot, \partial\Omega).$$

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