XVIII International Conference on Hyperbolic Problems
Theory, Numerics, Applications
Málaga, June 20-24, 2022

New Results for the Well-Posedness of Small BV Solutions to Systems of Conservation Laws

Geng Chen [*], William Golding [†], Sam G. Krupa [‡], Alexis F. Vasseur [§]

In this talk, I will present the following result from [7, 8],

Theorem 1 *For a system of conservation laws with two conserved quantities satisfying mild assumptions and endowed with a strictly convex entropy, solutions with small BV for each fixed time are unique and stable among the much larger class of weak solutions with large data, entropic for a strictly convex entropy and verifying a trace condition (which is weaker than BV_{loc}). The solutions with large data are also allowed to take values in the vacuum states.*

In particular, our result applies to the system of isentropic Euler equations for $\gamma > 1$. The result shows that the small BV solutions in Bressan-Crasta-Piccoli [2] are actually unique (and stable) among the the large class of weak entropic solutions.

Uniqueness of small BV solutions was established by Bressan and Goatin [3] under the *Tame Oscillation Condition*. Their work improved an earlier theorem by Bressan and LeFloch [4]. Uniqueness is also known when the Tame Oscillation Condition is replaced by the assumption that the trace of solutions along space-like curves has bounded variation (see Bressan and Lewicka [5]). This is the *Bounded Variation Condition*.

These uniqueness theories all need some a priori assumption on the solutions, such as the Tame Oscillation Condition or Bounded Variation Condition on space-like curves.

However, in this talk we show that solutions with small BV initial data (but possibly large data at positive times) and verifying a trace condition and an entropy condition for a strictly convex entropy, automatically satisfy the Bounded Variation Condition along space-like curves. Thus this a priori assumption is no longer needed.

Moreover, our result in this paper also gives stability. The existing L^1 stability theory for hyperbolic systems of conservation laws [1, 2, 6] considers the stability of solutions with small BV at each fixed time. In this L^1 stability theory, the perturbations must also be of small BV at each fixed time. In contrast, in the L^2 theory we use in this paper, the perturbation may be from the large class of weak solutions with possibly very large data. In this sense, our result gives a weak-BV stability result, similar to the weak-strong L^2 stability theory of Dafermos and DiPerna.

At a high level, the proof works within the framework developed first for scalar in [11]. The proof uses the techniques of the theory of shifts and a-contraction [9]. In the scalar case, we approximated a rough solution by a sequence of piecewise-smooth approximate solutions. In the scalar case, this allowed us to detect that the rough solution verified Oleĭnik's *condition E* and this put us in a class where we had uniqueness. In this talk, for the systems case, we approximate a rough solution with a sequence of piecewise-constant solutions constructed using a modified front tracking algorithm. This will allow us to detect the Bounded Variation Condition on the rough solution, and this will again put us in a class where we have uniqueness.

The basic idea of the proof of the theorem in this paper [7] is to consider a solution u with small BV initial data, but possibly very large and rough data for positive times, and then approximate the initial data with a piecewise-constant function. We can then use the front tracking algorithm to construct an approximate solution with this piecewise-constant initial data. To keep L^2

[*]University of Kansas. Email: gengchen@ku.edu
[†]The University of Texas at Austin. Email: wgolding@utexas.edu
[‡]Institute for Advanced Study. Email: skrupa@ias.edu
[§]The University of Texas at Austin. Email: vasseur@math.utexas.edu

stability between the front tracking solution and the rough solution u as time goes on, we must introduce shift functions into the local Riemann problems which are solved in the front tracking algorithm. We call this front tracking solution with shifts Ψ. Furthermore, in order for the shifts to keep L^2 stability, we must use a spatially inhomogeneous pseudo-distance. Thus, we must define a positive function $a(x,t)$ which acts as a weight on the space dimension x, where a is piecewise-constant function which only has a discontinuity at the places where the function $\Psi(x,t)$ has a shock. Thus, there are two difficulties in this paper: (a) the construction of the weight function a and (b) the introduction of shifts into the front tracking solution and the corresponding local Riemann problems.

We must construct a suitable weight function $a(x,t)$. For a point $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ where Ψ has a shock, we have constraints on the variations in the value of $a(x+,t) - a(x-,t)$ that depend on the size of the shock in Ψ and the family of the shock. Whenever there is an interaction between the waves in the function Ψ, the function a must be recomputed. The process of constructing the function a is one of the key difficulties because in order to get L^2 stability estimates on the growth in time of $\left\|u(\cdot,t) - \Psi(\cdot,t)\right\|_{L^2}$, the function a must be bounded away from zero.

In order to carefully construct a, we need refined quantitative control on how $\left|a(x+,t) - a(x-,t)\right|$ relates to the size of $\left|\Psi(x+,t) - \Psi(x-,t)\right|$ (the corresponding shock in Ψ). This refined control is proven in the companion paper [8], where it is shown that $\left|a(x+,t) - a(x-,t)\right|$ can be chosen proportional to $\left|\Psi(x+,t) - \Psi(x-,t)\right|$. This property was first shown in the class of inviscid limits of Navier-Stokes [10]. However, the proof based directly on the inviscid model is quite different, and very delicate.

Acknowledgements

G. Chen is partially supported by NSF with grants DMS-1715012 and DMS-2008504. W. Golding and S. Krupa were partially supported by NSF-DMS Grant 1840314. A. Vasseur is partially supported by the NSF grant: DMS 1614918.

References

[1] Alberto Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[2] Alberto Bressan, Graziano Crasta, and Benedetto Piccoli. Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws. *Mem. Amer. Math. Soc.*, 146(694):viii+134, 2000.

[3] Alberto Bressan and Paola Goatin. Oleinik type estimates and uniqueness for $n \times n$ conservation laws. *J. Differential Equations*, 156(1):26–49, 1999.

[4] Alberto Bressan and Philippe LeFloch. Uniqueness of weak solutions to systems of conservation laws. *Arch. Rational Mech. Anal.*, 140(4):301–317, 1997.

[5] Alberto Bressan and Marta Lewicka. A uniqueness condition for hyperbolic systems of conservation laws. *Discrete Contin. Dynam. Systems*, 6(3):673–682, 2000.

[6] Alberto Bressan, Tai-Ping Liu, and Tong Yang. L^1 stability estimates for $n \times n$ conservation laws. *Arch. Ration. Mech. Anal.*, 149(1):1–22, 1999.

[7] Geng Chen, Sam G. Krupa, and Alexis F. Vasseur. Uniqueness and weak-BV stability for 2×2 conservation laws. *arXiv e-prints*, page arXiv:2010.04761, October 2020.

[8] William Golding, Sam Krupa, and Alexis Vasseur. Sharp a-contraction estimates for small extremal shocks, 2021.

[9] Moon-Jin Kang and Alexis F. Vasseur. Criteria on contractions for entropic discontinuities of systems of conservation laws. *Arch. Ration. Mech. Anal.*, 222(1):343–391, 2016.

[10] Moon-Jin Kang and Alexis F. Vasseur. Uniqueness and stability of entropy shocks to the isentropic Euler system in a class of inviscid limits from a large family of Navier-Stokes systems. *arXiv:1902.01792*, 2020. To appear in Inventiones Mathematicae.

[11] Sam G. Krupa and Alexis F. Vasseur. On uniqueness of solutions to conservation laws verifying a single entropy condition. *J. Hyperbolic Differ. Equ.*, 16(1):157–191, 2019.