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Convex integration applied to the compressible Euler equations.

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1 Introduction

In this talk we consider the barotropic compressible Euler equations

$$\begin{aligned} \partial_t \varrho + \operatorname{div} \mathbf{m} &= 0\,, \\ \partial_t \mathbf{m} + \operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p(\varrho)\right) &= \mathbf{0}\,, \end{aligned} \tag{1}$$

in two or three space dimensions with unknown density $\varrho = \varrho(t, \mathbf{x}) \in \mathbb{R}^+$ and momentum $\mathbf{m} = \mathbf{m}(t, \mathbf{x}) \in \mathbb{R}^n$ (with $n = 2$ or $n = 3$), which are both functions of time $t \in [0, \infty)$ and position $\mathbf{x} \in \mathbb{R}^n$. The pressure $p = p(\varrho)$ is a given function of ϱ.

In *one* space dimension the question of well-posedness for the compressible Euler equations has been answered quite satisfactory: Glimm [8] showed existence of BV weak entropy[1] solutions for sufficiently small initial data and Bressan et al. [1] proved uniqueness of these solutions in BV class. But in *multiple* space dimensions it has been shown in the past 15 years, that solutions are not unique in the class of bounded weak entropy solutions. These non-uniqueness results are achieved using a technique called *convex integration*, which was developed among others by Gromov in the context of partial differential relations and is built upon Nash's proof of the isometric embedding problem.

2 Overview over some non-uniqueness results for compressible Euler

After the introduction we will briefly summarize some of the results on convex integration for system (1) which are available in the literature:

1. Convex integration was first used in the context of fluid flow equations by De Lellis and Székelyhidi [5], who applied it to the *incompressible* Euler equations. This led to infinitely many solutions to the incompressible Euler system, infinitely many of which are even *pressureless*.

   The first result on non-uniqueness for *compressible* Euler (1) (see De Lellis, Székelyhidi [6]) is a simple consequence by setting $\varrho \equiv 1$ and $\mathbf{m} = \mathbf{v}$, where $\mathbf{v}$ is such a pressureless incompressible solution.

2. Later Chiodaroli [2] as well as Feireisl [7] showed independently that for any continuously differentiable initial density ϱ_0 there exists bounded initial momentum for which there are infinitely many weak entropy solutions to the compressible Euler system (1).

   Chiodaroli's ansatz is to look for solutions with constant-in-time density $\varrho(t, \cdot) = \varrho_0$, $\forall t$. Feireisl's idea is to apply Helmholtz decomposition to the momentum, i.e. to write $\mathbf{m} = \mathbf{v} + \nabla\Phi$ where $\mathbf{v}$ is div-free and Φ is a scalar field. He then prescribes ϱ and Φ such that they are compatible with the conservation of mass, i.e. $\partial_t \varrho + \Delta\Phi = 0$ and applies convex integration to find appropriate $\mathbf{v}$.

3. The literature also provides non-uniqueness results for compressible Euler with Riemann initial data, see [3, 4, 9]. These solutions are constructed with piecewise constant density.

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[1] Note that in the context of the Euler system (1), a weak *entropy* solution is a weak solution which satisfies the *energy* inequality.

3 Basic ideas of the convex integration technique

Finally we present the basic ideas of how the convex integration technique works. The first step is to rewrite the Euler equations (1) as a differential inclusion: One considers

$$\begin{aligned} \partial_t \varrho + \operatorname{div} \mathbf{m} &= 0, \\ \partial_t \mathbf{m} + \operatorname{div} (\mathbb{U} + q\mathbb{I}) &= \mathbf{0}, \end{aligned} \tag{2}$$

where $\mathbb{U}$ and q are additional unknown functions which take values in the symmetric traceless matrices and in $\mathbb{R}$, respectively. Instead of solving (1) one can equivalently look for solutions of the underdetermined linear system (2) which take values in the set

$$K := \left\{ (\varrho, \mathbf{m}, \mathbb{U}, q) \,\middle|\, \mathbb{U} + q\mathbb{I} = \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho) \right\}.$$

Now a *subsolution* is defined as a solution of the linear system (2) which takes values in the Λ-convex hull K^Λ of the set K, where Λ is the wave cone which corresponds to (2). Roughly speaking convex integration is a method which forms subsolutions into solutions by adding plane wave oscillations.

With this method at hand, it suffices to find an appropriate subsolution if one wants to solve (1). In particular, convex integration becomes a powerful tool if the class of possible subsolutions is large, i.e. if K^Λ is much larger than K.

In this talk we explain two convex integration approaches for the compressible Euler system (1):

1. The first one is used in all results mentioned in Section 2. The key idea is is to consider the density ϱ as well as q only as parameters. First these parameters are fixed, then a slight modification of De Lellis' and Székelyhidi's incompressible convex integration is applied to construct $\mathbf{m}$ and $\mathbb{U}$.

   This approach seems to have a weak point: The fact that ϱ and q are merely parameters might restrict the class of possible subsolutions and hence make the technique less powerful than a genuine compressible approach.

2. In order to tackle this weak point, the speaker has developed the first genuinely compressible convex integration, see [10]. However this does *not* lead to a larger class of possible subsolutions.

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