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Stumpons are non-conservative traveling waves of the Camassa–Holm equation

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The Camassa–Holm equation, which for $u = u(t, x)$ and $t, x \in \mathbb{R}$ reads

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1}$$

has been studied in great detail since it was derived in [1] in the context of shallow water waves. Much of this attention can be explained by its many appealing mathematical properties; for instance, it is completely integrable, bi-Hamiltonian and thus admits infinitely many conserved quantities. One of these invariants is

$$\int_{\mathbb{R}} \left(u^2 + u_x^2\right) \mathrm{d}x, \tag{2}$$

and this is typically called the energy of the equation. Moreover, it has a number of traveling wave solutions, cf. [2], the most famous of which are the peakons with their explicit formulas. We note that since traveling waves are translations of their initial profiles, it is clear that they preserve quantities such as the energy (2).

Yet another interesting feature of (1) is that it admits a singularity formation known as wave breaking, i.e., initially smooth wave profiles may steepen such that for some x, the slope $u_x(\cdot, x)$ becomes unbounded from below in finite time. This may lead to a concentration of energy at the point of wave breaking, where $u_x^2 \, \mathrm{d}x$ in (2) tends to a singular measure. As a consequence, there is an ambiguity in how to extend the solutions of (1) past the time of wave breaking, since we can remove some or all of the concentrated energy and still have a weak solution. In order to resolve this problem, the notion of conservative solutions has been introduced in [3, 4] by imposing the additional requirement that the solution weakly satisfies an additional conservation law which for sufficiently regular u reads

$$(u^2 + u_x^2)_t + (u(u^2 + u_x^2))_x = (u^3 - 2Pu)_x, \tag{3}$$

where P satisfies $P - P_{xx} = u^2 + \frac{1}{2}u_x^2$. In general, one replaces $u^2 + u_x^2$ in (3) with a measure μ where the absolutely continuous part satisfies $\mathrm{d}\mu_{\mathrm{ac}} = (u^2 + u_x^2)\, \mathrm{d}x$. A consequence of this requirement is that conservative solutions preserve their initial energy (2), even past wave breaking.

The Camassa–Holm equation has several two-component extensions to systems, and the recent paper [5] presented a discretization based on variational principles in Lagrangian coordinates for such a system, designed to approximate conservative solutions. A periodic version of this discretization was implemented numerically in [6], and its performance was compared to existing numerical methods for (1). The method produced good results, even when approximating reference solutions with wave breaking. This raised the question whether the discretization also could handle the traveling wave solutions known as stumpons, which have been troublesome to approximate with numerical methods. For appropriate parameter values these stumpons can be constructed from cusped traveling waves, known as cuspons, by adding a plateau of constant height to their crest.

It turned out that, unlike for cuspons, the scheme in [6] was not able to approximate stumpons in a satisfactory manner, irrespective of the degree of refinement in the discretization, cf. Figure 1. This motivated a more thorough study of the stumpon solutions, and we found that stumpons, unlike peakons and cuspons, do not satisfy (3) in the weak sense. To be specific, it fails to satisfy a Rankine–Hugoniot-type condition at the boundary points of the plateau. Furthermore, by considering the equation in Lagrangian

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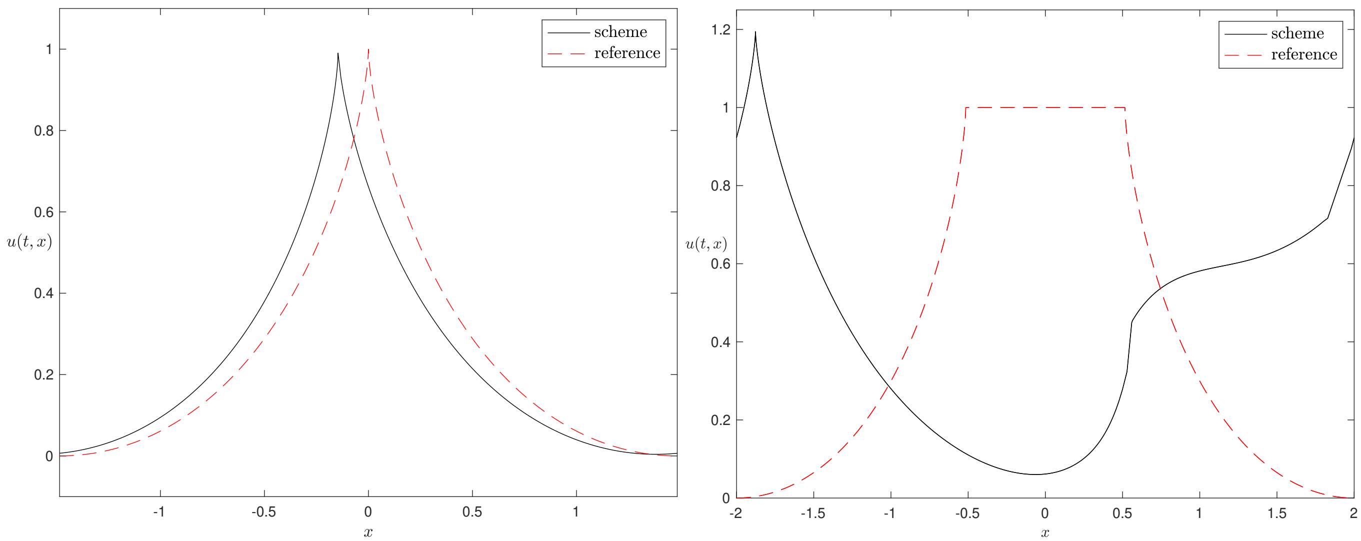

(a) (b)

Figure 1: Conservative numerical approximations and reference solutions after two periods for initial data given by a cuspon (a) and a stumpon (b).

coordinates and employing techniques from [7], we prove that the initial behavior exhibited by the numerical approximation, i.e., the collapse of the plateau, is exactly what is expected from a conservative solution. These findings are reported in [8].

In conclusion, the fact that stumpons are not conservative could explain why numerical methods which manage to resolve peakons and cuspons have problems with stumpons. In particular, it seems that to approximate stumpons one has to design a numerical method based on different principles altogether.

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