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A stochastic fluid-structure interaction problem describing Stokes flow interacting with a membrane

Jeffrey Kuan [*], Suncica Canic[†]

We study a stochastic fluid-structure interaction (FSI) problem, describing the dynamical interaction between a viscous, incompressible fluid and an elastic structure, under the additional influence of stochastic noise in time acting on the structure. We model the fluid by the time dependent linear Stokes equation, and we model the elastic structure by the wave equation. We establish existence and uniqueness of appropriately defined weak solutions to this stochastic FSI problem by developing a new splitting scheme, which gives approximate solutions to the stochastic FSI problem, which are random. We then use compactness arguments and probabilistic methods to pass to the limit and obtain existence of solutions.

Fluid-structure interaction (FSI), which describes the coupled dynamical interaction between a fluid and solid structure, has applications to many disciplines, including civil, biomedical, and mechanical engineering. Despite its importance, FSI problems are mathematically challenging due to the complex coupling between the fluid and structure. The development of robust methods for analyzing FSI is an area of active research. While there has been extensive past work on deterministic FSI systems (see e.g. [2, 3, 5]), the rigorous analysis of stochastic fluid-structure interaction has been largely unexplored. Though a wide variety of stochastic PDEs have been considered in past literature, the study of stochastic fluid-structure interaction systems, involving coupled fluid-solid PDEs with stochastic noise, has remained open. The current work fills this gap by providing the first well-posedness result for a fully coupled stochastic fluid-structure interaction system. Such models are important in applications of FSI, where random noise in the system can affect the resulting coupled fluid-structure dynamics. An example of such a problem is the flow of blood in coronary arteries that sit on the surface of the heart and contract and expand under the outside forcing due to the heart muscle contraction and expansion. Dynamic patient images show significant stochastic effects in the heart contractions that can be captured by an FSI model such as the one studied in this work.

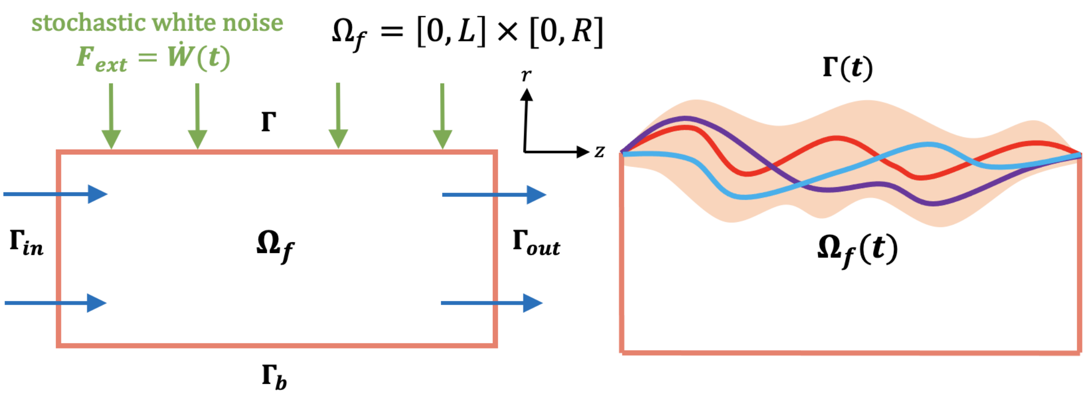

Figure 1: Left: The domain for the linearly coupled stochastic FSI problem, where a stochastic forcing F_{ext} in the form of white noise in time acts on the structure. Right: An illustration of the time-dependent fluid domain and structure configuration, which are random. The different colors represent different possible outcomes for the random configuration of $\Gamma(t)$. The lightly colored region represents a confidence interval of where $\Gamma(t)$ is expected to be.

[*]Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720. Email: jeffreykuan@berkeley.edu
[†]Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720. Email: canics@berkeley.edu

The model we consider involves a fluid residing in a two-dimensional rectangular domain $\Omega_f = [0, L] \times [0, R]$, which represents half of a full rectangular domain, with a line of even symmetry at the bottom boundary of Ω_f (see the left panel of Fig. 1). The top boundary of Ω_f is the reference configuration Γ of the elastic structure, which we will assume displaces in only the radial direction. We emphasize that we are first considering a linearly coupled model, so that the fluid domain is assumed to be fixed in time. This is a linearization of the more general nonlinearly coupled case, in which the displacement of the elastic membrane determines the time-dependent moving fluid domain (see the right panel of Fig. 1).

We model the fluid, which resides in Ω_f, by the linear Stokes equations

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{u} - \mu\Delta \boldsymbol{u} + \nabla p &=& 0, \\ \nabla \cdot \boldsymbol{u} &=& 0, \end{array}\right\} \quad \text{in } \Omega_f. \tag{1}$$

The structure, whose reference configuration is given by Γ, is modeled by the wave equation for the radial displacement $\eta(t, z)$:

$$\partial_{tt}\eta - \Delta\eta = f, \qquad \eta(t, 0) = \eta(t, L) = 0, \tag{2}$$

where f is a source term that will be specified later.

We couple the fluid and structure using two coupling conditions. The *kinematic coupling condition* is a no-slip condition that states that $\boldsymbol{u} = \partial_t \eta \boldsymbol{e}_r$ on Γ, where $\boldsymbol{e}_r$ is the unit vector in the radial direction. The *dynamic coupling condition* describes the contribution of the fluid and other external forces on the elastodynamics of the structure and specifies the source term f in (2) so that the structure equation is

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma}\boldsymbol{e}_r \cdot \boldsymbol{e}_r + F_{ext}, \ \text{ on } \Gamma, \ \text{with } F_{ext} = \dot{W}(t),$$

where F_{ext} denotes external forcing on the structure. To consider stochastic effects, the outside forcing on the membrane will be represented by a temporal white noise $\dot{W}(t)$ which is constant in space, where $\dot{W}(t)$ is white noise in time and $W(t)$ is a one-dimensional Brownian motion. Although one can consider more general types of random noise, this simple case is a prototypical case which demonstrates many of the main difficulties.

We show **existence and uniqueness of a solution to this stochastic FSI problem** by using a Lie operator splitting which splits the problem into a structure, stochastic, and fluid subproblem, and combines the stochastic and deterministic splitting strategies from [5] and [1]. The resulting splitting scheme is used to semi-discretize the problem in time and, at every time step, solve a sequence of subproblems defined by the splitting scheme in order to construct approximate solutions, which we emphasize are random variables. Even though the resulting coupled problem is linear, we must use compactness arguments and probabilistic methods, such as the Skorokhod representation theorem and the Gyöngy-Krylov theorem [4], to pass to the limit and obtain existence of the solution. This is because uniform energy estimates can be obtained only *in expectation*. So, passing to the limit requires first proving convergence of the corresponding sequence of probability measures given by the laws of the approximate solutions, from which one can deduce almost sure convergence along a subsequence of the approximate solutions by using the Skorokhod representation theorem and the Gyöngy-Krylov theorem.

This work is significant because it develops a general framework that can be used to analyze complex stochastic systems with coupling, such as stochastic fluid-structure interaction systems.

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