XVIII International Conference on Hyperbolic Problems
Theory, Numerics, Applications
Málaga, June 20-24, 2022

On gradient flow and entropy solutions for nonlocal transport equations with nonlinear mobility

Simone FAGIOLI *, Oliver TSE †

We prove the well-posedness of entropy solutions for a wide class of nonlocal transport equations with nonlinear mobility in one spatial dimension,

(1) $$\partial_t \rho_t = \partial_x \left(\vartheta(\rho_t)\, \partial_x \mathcal{F}'(\rho_t)\right) \qquad \text{in } (0,\infty)\times\mathbb{R}\,,$$

where $\vartheta : [0,\infty) \to [0,\infty)$ is a given *mobility* function, and $\mathcal{F}$ is a driving energy functional of the form

$$\mathcal{F}(\rho) = \int_{\mathbb{R}} V(x)\rho(x)\,dx + \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} W(x-y)\rho(x)\rho(y)\,dx\,dy\,,$$

with external potential $V : \mathbb{R} \to \mathbb{R}$ and interaction potential $W : \mathbb{R} \to \mathbb{R}$. When $\vartheta(\rho) = \rho$, the equation reduces to a nonlocal transport equation, and (1) is known to possess a 2-Wasserstein gradient flow structure, and the, by now standard, AGS theory for gradient flows in metric spaces may be employed to study such equations for a general class of driving functionals $\mathcal{F}$ [1]. When ϑ is nonlinear, equation (1) can be derived from models that often appear in the transport phenomena of biological systems with overcrowding prevention , and in the studies of phase segregation, relaxation of fermionic gas and vortex formation in mathematical physics . In contrast to the linear case, the geometric nature of (1) is less understood when ϑ is nonlinear. For example, when $V(x) = -x$ and $W \equiv 0$, (1) reduces to the scalar conservation law

$$\partial_t \rho_t + \partial_x \vartheta(\rho_t) = 0\,,$$

where ϑ now plays the role of the flux function. Therefore, providing a gradient flow structure to (1) is akin to providing a gradient flow structure for a scalar conservation law. In [2], we shed new light onto the unveiling of a gradient flow structure for (1), but also provide the well-posedness (in the sense of entropy solutions) of (1) for a general class of interaction potentials, including the Newtonian potential. This route is inspired by deterministic particle approximations (DPA) designed for equations like (1) [3], by recent advances in the theory of generalized gradient structures [5], and by the asymptotic limits of such structures [4]. More explicitly, we introduce a DPA for the approximation of (1) that possesses a generalized gradient structure, and show that this approximation converges to the unique entropy solution of (1), which also exhibits a gradient structure. This choice of approximation restricts our analysis to the 1-dimensional setting, because its generalization to higher dimensions is not straightforward.

References

[1] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[2] S. Fagioli, and O. Tse. On gradient flow and entropy solutions for nonlocal transport equations with nonlinear mobility *Submitted preprint*, 2021.

[3] M. Di Francesco, S. Fagioli, and E. Radici. Deterministic particle approximation for nonlocal transport equations with nonlinear mobility. *J. Differential Equations*, 266(5):2830–2868, 2019.

[4] A. Mielke. On evolutionary Γ-convergence for gradient systems. In *Macroscopic and large scale phenomena: coarse graining, mean field limits and ergodicity*, volume 3 of *Lect. Notes Appl. Math. Mech.*, pages 187–249. Springer, [Cham], 2016.

[5] M. A. Peletier, R. Rossi, G. Savaré, and O. Tse. Jump processes as generalized gradient flows. *Calc. Var. Partial Differential Equations*, 61(1):Paper No. 33, 2022.

*Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica Università degli Studi dell'Aquila, Via Vetoio 1, 67100 Coppito, L'Aquila, It. Email: simone.fagioli@univaq.it

†Department of Mathematics and Computer Science, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands. Email: o.t.c.tse@tue.nl