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Weakly Compressible Two-layer Shallow-Water Flows with Friction and Entrainment along Channels

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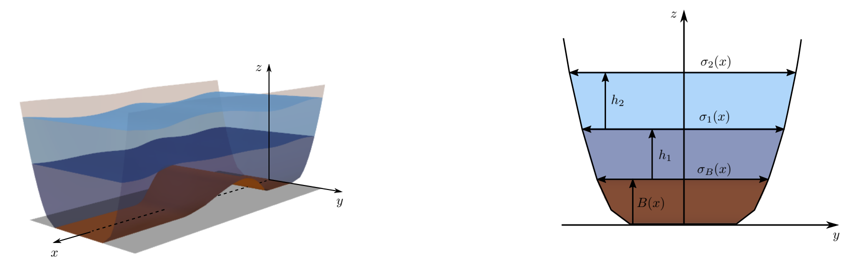

(a) (b)

Figure 1: Schematic of a two layer channel flow with the two layers moving in opposite directions: (a) Full 3D view of the flow, (b) channel cross section.

We present a weakly compressible approach to describe two-layer shallow water flows in channels with a bottom topography and arbitrary cross sections. See the schematic in Figure 1. These flows are characterized by a large horizontal length scale relative to their depth. Applications of this model include the study of internal waves observed in rivers and possibly caused by shear flow instabilities [2]. Shallow-water flows are typically modeled by the Saint-Venant equations, a hyperbolic balance law that results from the *depth averaging* of the Euler equations.

Computations of two layer flows in channels are more challenging than those occurring in one layer. Theoretical difficulties arise due to the presence of non-conservative products. For instance, the standard approach for those flows results in a conditionally hyperbolic balance law with non-conservative products. See [5] for more details. In this work, we consider the fluid to be weakly compressible and follow a relaxation approach as in [1]. The corresponding model is unconditionally hyperbolic. See [4] for a related work in channels with vertical walls and constant width. In the talk, a detailed description of the properties of the model will provided, including entropy inequalities. Furthermore, numerical experiments using a central-upwind scheme will be presented.

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1 Details of Model

The model for the incompressible two-layer shallow water flows in channels can be found in [3]. After adding source terms due to friction and entrainment and considering the weak compressibility of both layers, it has been re-written as

$$\frac{\partial(\rho_1 A_1)}{\partial t}+\frac{\partial(\rho_1 Q_1)}{\partial x}=\rho_1 S_e, \tag{1a}$$

$$\frac{\partial A_1}{\partial t}+u_1\frac{\partial A_1}{\partial x}=\frac{\mu\left(p_1-p_o-g\int_B^{w_1}(w_1+rh_2-z)\sigma(x,z)\right)dz}{\rho_1 c_1^2} \tag{1b}$$

$$\frac{\partial(\rho_1 Q_1)}{\partial t}+\frac{\partial}{\partial x}\left(\rho_1 A_1 u_1^2+\rho_1 p_1\right)=g\rho_1\left(I_1-h_1\sigma_B B'+rh_2\frac{\partial A_1}{\partial x}\right)+\rho_1 S_{f,1}+\rho_1 S_e u_1,+p_o\frac{\partial \rho_1}{\partial x} \tag{1c}$$

$$\frac{\partial(\rho_2 A_2)}{\partial t}+\frac{\partial(\rho_2 Q_2)}{\partial x}=-r\rho_2 S_e, \tag{1d}$$

$$\frac{\partial A_2}{\partial t}+u_2\frac{\partial A_2}{\partial x}=\frac{\mu\left(p_2-p_o-g\int_{w_1}^{w_2}(w_2+rh_2-z)\sigma(x,z)\right)dz}{\rho_2 c_2^2} \tag{1e}$$

$$\frac{\partial(\rho_2 Q_2)}{\partial t}+\frac{\partial}{\partial x}\left(\rho_2 A_2 u_2^2+\rho_2 p_2\right)=g\rho_2\left(I_2-h_2\sigma_1\frac{\partial w_1}{\partial x}\right)+\rho_2 S_{f,2}-r\rho_2 S_e u_2+p_o\frac{\partial \rho_2}{\partial x}. \tag{1f}$$

Here h_1, and h_2 denote the depth of each layer, u_1, and u_2 the the cross-sectional velocities; g the acceleration of gravity, $B(x)$ describes the bottom topography, and $\sigma(x,z)$ the width of the channel; $A_1=\int_B^{w_1}\sigma(x,z)\,dz$, and $A_2=\int_{w_1}^{w_2}\sigma(x,z)\,dz$ are the cross-sectional *wet* areas in each layer; $w_1=B+h_1$ denotes the total elevation of the internal layer and $w_2=B+h_1+h_2$ that of the external layer; and $Q_1=A_1u_1$, and $Q_2=A_2u_2$ are the flow rates or discharges for the internal and external layers respectively. Furthermore, $\sigma_B(x)=\sigma(x,B(x)),\sigma_1(x,t)=\sigma(x,w_1(x,t)),\sigma_2(x,t)=\sigma(x,w_2(x,t))$ denote the channel width at the bottom topography, and at the internal and external layers respectively. We note that σ_1 and σ_2 depend both on space and time since they are evaluated at the internal and external layers. The ratio of densities is denoted by $r=\rho_2/\rho_1\leq 1$. The relaxation parameter is given by μ. The vertically integrated hydrostatic pressures of both layers are given by

$$p_2=p_o+{c_2}_o^2(\rho_2-\rho_2^o)+g\int_{w_1}^{w_2}(w_2-z)\,\sigma(x,z)\,dz,\;\; p_1=p_o+{c_1}_o^2(\rho_1-\rho_1^o)+g\int_B^{w_1}(w_1+rh_2-z)\,\sigma(x,z)\,dz. \tag{2}$$

The source terms I_1, and I_2 correspond to the vertically integrated pressure terms due to width variation, and are given by

$$I_1=I_1(x,t)=\int_B^{w_1}(w_1-z)\,\sigma_x(x,z)\,dz,\qquad\text{and}\qquad I_2=I_2(x,t)=\int_{w_1}^{w_2}(w_2-z)\,\sigma_x(x,z)\,dz. \tag{3}$$

The friction for each layer, and entrainment terms are given by $S_{f,1}$, $S_{f,2}$, and S_e respectively. The speed of sound for both layers are respectively given by

$$c_1=\sqrt{g\left(\epsilon+r\frac{\sigma_1}{\sigma_2}\right)\frac{A_1}{\sigma_1}}=\sqrt{g\left(r\frac{A_1}{\sigma_2}+\epsilon\frac{A_1}{\sigma_1}\right)},c_2=\sqrt{\frac{gA_2}{\sigma_2}}, \tag{4}$$

where $\epsilon=1-r$.

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