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# Time-dependent eikonal equation with Soner boundary condition

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A robust numerical algorithm to compute a time-dependent eikonal equation for the signed or wall distance function on three dimensional polyhedral meshes is an important component to use the level set method [1, 2, 3] in many industrial applications. In numerical combustion, the signed distance function whose zero level set represents a thin flame surface is used to model the $G$-equation and evolve the mean flame surface in a turbulent flow. It is also used to accurately model the flame-wall interaction and quenching or the end-gas autoignition for knock prediction. In computational fluid dynamics (CFD), the level set method related to two-phase flow problems or front capturing applications needs to keep the property of the signed distance from the interface. The wall distance function has been a key feature of turbulent modeling in CFD. There is an extensive amount of literature that discusses a further usage of signed distance functions; surface reconstruction in computer graphics, meshing and medial axis transformation, boundary and spatial-partitioning representations for the navigation of robots, heterogeneous material modeling, automatic removal of geometrical shape feature in computer-aided design.

In the view of numerical algorithms to compute the signed or wall distance function, each algorithm has been developed by different purpose and perspective to use the distance properties in its main application. It diversifies the development of numerical algorithms into various methods to focus on numerical properties such as computational cost, efficiency, accuracy, robustness, and parallelization. In this talk, a numerical algorithm is introduced to focus on its robustness for industrial applications with reasonable requirements [13]. Due to the size and complexity of the shape of the computational domain, polyhedral meshes and parallel computing are mostly preferred to capture the reality corresponding the simulation; see more details in the reference [4]. In order to implement a new numerical algorithm straightforwardly into the prevalent CFD numerical codes, a vertex-centered or cell-centered finite volume method is a preferable choice.

Numerical methods to compute a time-dependent eikonal equation for signed or wall distance function is identical to numerically find the viscosity solution of the eikonal equation:

$$\begin{aligned} |\nabla\phi(\mathbf{x})| &= 1, \quad \mathbf{x}\in\Omega \\ sgn(\phi) &= sgn(\phi_0), \end{aligned} \tag{1}$$

where $sgn(x)$ is a sign function on $\mathbb{R}$, $\Omega$ is the computational domain, and $\phi_0(\mathbf{x})$ is a continuous function on $\Omega$ whose zero level set is the given surface in $\bar{\Omega}$. In this talk, we show the bidirectional time-relaxed (time-dependent) eikonal equation, called the bidirectional flow equation [11, 12], to numerically obtain the signed distance function from an evolving surface on a whole computational domain discretized by polyhedral meshes. A practical usage of level set method in industrial applications like combustion, multiphase flow, and the other front capturing problems requires computational domains of very general shapes. The robustness and accuracy of the numerical algorithm is more important than the computational cost as long as a parallel computing is reasonably possible. Additionally, we would like to cover the case that the given surface is a part of the boundary of the domain. A typical application of the mentioned case is path planning, visibility detection, optimal control, and shape-from-shading; see more details in the reference [1, 2, 3].

In the case of general shape of the computational domain, we explain that the Soner boundary condition, called state-constraint condition [5, 6, 7, 8] in the shape-from-shading, should be used to numerically approach the viscosity solution of the eikonal

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equation. The incompleteness of measured data [5] is the main cause to consider the Soner boundary condition in shape-from-shading. Similarly, the lack of the visibility from the given surface on a non-convex domain is the main reason to apply the same condition in the case of computing the signed distance function on a non-convex domain. The Soner boundary condition is called as the no-inflow boundary condition [9]. The similar concept of the boundary condition is also considered in the reinitialization close to contact lines in CFD [10].

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