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High-order accurate bound-preserving space-time limiting for conservation laws

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We provide a framework for high-order accurate discretizations of the general scalar nonlinear conservation law

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f}(u) - \nabla \cdot (c(u, \mathbf{x}) \nabla u) = 0, \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \quad d \in \{1, 2, 3\}, \tag{1}$$

with $c(u, \mathbf{x}) \geq 0$. In many applications, it is important to preserve a global maximum or minimum principle:

$$\min_j u^n \leq u_i^{n+1} \leq \max_j u^n. \tag{2}$$

For instance, u may represent a volume fraction, in which case it should remain in the interval $[0, 1]$. It is notoriously difficult to ensure satisfaction of (2) when solving (1) with a high-order numerical discretization. We propose a general framework to achieve this based on combining a first-order numerical solution that is guaranteed to satisfy (2) with a high-order solution that may violate (2). This general approach has been applied in many existing schemes developed over the last few decades. Of course, the key is to combine the two solutions in a way that preserves high-order accuracy as much as possible while enforcing (2).

Although our framework is applicable to a range of numerical discretizations, we present it here within a finite volume framework. A suitable low-order method can be obtained using the local Lax-Friedrichs flux for the convective term:

$$F_{ij}^L(u) = \mathbf{n}_{ij} \cdot \frac{\mathbf{f}(u_j) + \mathbf{f}(u_i)}{2} - \frac{1}{2}(u_j - u_i)\lambda_{ij}^A. \tag{3}$$

Here u_j denotes the average of the solution over cell j, and F_{ij}^L is the low-order numerical flux. The diffusive flux is taken as

$$P_{ij}^L(u) = \underbrace{c\left(\frac{u_i + u_j}{2}, \frac{\mathbf{x}_i + \mathbf{x}_j}{2}\right)}_{=: \, c_{ij}} \frac{u_j - u_i}{|\mathbf{x}_j - \mathbf{x}_i|}. \tag{4}$$

Discretizing in time with the backward Euler method, the resulting scheme can be shown to satisfy (2), but is only first order accurate in time and space.

We formulate a high-order discretization of (1) using weighted essentially non-oscillatory (WENO) reconstruction in space and Runge-Kutta integration in time. For details, we refer to [1]. The resulting scheme is high-order accurate and reasonably well-behaved (due to the WENO limiting) but does not satisfy (2) in general.

The composite scheme using both methods can be written

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \mathcal{N}_i} |S_{ij}| \left[G_{ij}^L - \alpha_{ij}(G_{ij}^L - G_{ij}^H) \right], \tag{5}$$

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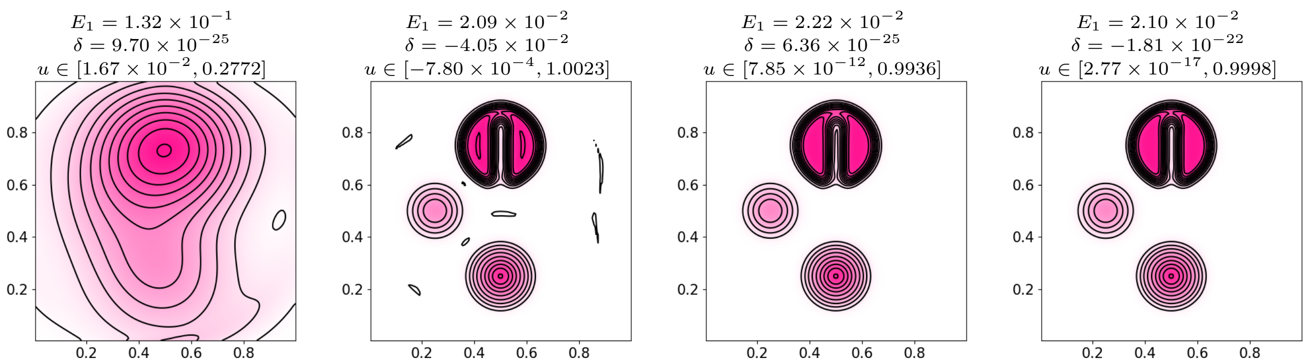
where G^L and G^H denote low- and high-order fluxes, respectively; K_i and $|S_{ij}|$ are the cell volumes and face areas, respectively, and the summation is over all cells neighboring cell i. The values α_{ij} are referred to as limiters and will be chosen in order to enforce (2).

We consider two methods for choosing the limiter values; the first is based on flux-corrected transport, while the second is referred to as global monolithic convex limiting, first developed in [2]. For details, we refer to [1].

The novelty of our approach lies in applying limiters to the full space-time discretization. In contrast to the traditional strong stability preserving (SSP) approach, our limiters enable the use of general Runge-Kutta schemes and can be arbitrarily high order accurate in time. If required, the limiters can be applied to each Runge-Kutta stage; alternatively, they can be applied only to the final update in the computation of each step. We show that there exists a class of Runge-Kutta method of arbitrarily high order for which (2) is guaranteed at the intermediate stages when limiting is applied only in space; for these methods, applying space-time limiting to only the step update is always sufficient.

We test our approach on several numerical examples, including the solid rotation problem shown in Figure 1, as well as various convection-diffusion problems. The solution obtained with either limiting method is bound-preserving for arbitarily large step sizes and exhibits high-order convergence for smooth solutions.



(a) Low-BE (b) WENO-SDIRK (c) FCT-SDIRK (d) GMC-SDIRK

Figure 1: Numerical solution of the solid rotation problem using low-order, unlimited, FCT, and GMC schemes. In all cases, we use $N_h = 128^2$ uniform square cells. The color scale in the plots goes from white to pink, which corresponds to 0 and 1, respectively.

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References

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