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Multiresolution-analysis for stochastic hyperbolic conservation laws*

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We discuss random hyperbolic conservation laws with uncertain initial data. It is well known, that for deterministic hyperbolic conservation laws discontinuities may arise even in simple problems over time. Considering random hyperbolic conservation laws, the discontinuities that arise in spatial directions can also affect the stochastic behavior, leading to discontinuities along the stochastic direction. This makes it difficult to treat such problems. To study the interplay between the spatial and the stochastic directions, we interpret the stochastic as an additional spatial parameter of the original problem. This results in a new higher dimensional deterministic problem. To justify our higher dimensional approach we have proven in [5] that for absolute continuous random variable its solution coincides with the stochastic solution described in [1]. In particular, we have proven the existence of stochastic moments when the stochastic moments exist for the uncertain initial value.

The solution for the novel approach is approximated by a DG-scheme. In order to determine the stochastic moments, we additionally apply multi-element stochastic collocation methods to the numerical solution of the deterministic problem. Since we have a higher dimensional problem, it is computationally expensive to perform the simulation on a fully refined grid. Thus, we apply grid adaptation which allows to compute an efficient solution preserving the accuracy of the uniform scheme. In particular, we use grid adaptation based on multiresolution analysis, see for example [2, 3], which belongs to the class of perturbation methods. Thus, we calculate the multiscale decomposition of a solution to study its detail coefficient describing its local contribution to a cell. Due to the cancellation property [4] the details of the solution are locally small when the solution is locally smooth and locally large when the solution has high local changes like discontinuities. This allows us to distinguish between significant and non-significant cells and we can neglect cells whose detail coefficients are below a certain threshold value, whereas regions with high locale changes have large detail coefficients that cannot be neglected. Thus, refinement in regions with high local changes results in an adaptive grid with significantly fewer cells than a uniform grid. Therefore, we are able to detect discontinuities in both the stochastic and the spatial direction.

In the "uniform thresholding" strategy described above, grid adaptation only considers the data of the solution itself, so the stochasticity has no effect on the adaptation at all. Since we are interested in stochastic moments rather the solution itself, grid adaptation is not optimal for this purpose and we prefer an adaptation strategy based on the stochastic moments of the solution. In our work, we study multi-resolution analysis based on the stochastic moments of the solution to derive an adaptation strategy taking into account the weight function. To account for the stochasticity, we modify the threshold procedure to include the corresponding probability density. This results in a grid adaptation with "weighted thresholding". Thus, grid adaptation depends on the local contribution of the probability density and triggers more grid refinement in regions with high local stochastic influence, whereas regions with no stochasticity at all are attenuated in the grid adaptation. Using this novel adaptation strategy, we are able to proof that the resulting perturbation error between the stochastic moments of the approximation of the solution on a uniform grid and the stochastic moments on its sparse representation using our novel strategy can be bounded by a global threshold value. Moreover, we show that the L^1-error of the stochastic moments using weighted thresholding is bounded by the discretization error of the solution and the perturbation error to its sparse representation using weighted thresholding. Assuming a converging DG-scheme, we obtain L^1-convergence of our novel adaptation strategy.

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To verify our theoretical results, we have performed some numerical examples comparing the L^1-error of the stochastic moments using uniform thresholding and weighted thresholding. In Figure 1 we see the solution of the Burgers' equation with smooth uncertain initial data $u_0(x,\xi) = \sin(2\pi x)\sin(2\pi\xi)$, $(x,\xi) \in [0,1]^2$. Here, the uncertainty is described by a beta distributed random variable $\mathcal{B}(2,20)$, which has a high probability density for $\xi \in [0,0.2]$. Furthermore, we show the adaptive grids for some final time using uniform thresholding and weighted thresholding, respectively. Compared to uniform thresholding, our novel method triggers more grid refinement in regions with a high local contribution of the probability density, while regions without stochastic influences are not refined at all. This leads to an adaptive grid with respect to the stochastic behavior. Although, the solution itself may look poor using weighted thresholding, we see in Figure 1 that the L^1-error of the stochastic moments is comparable to the L^1-error when using uniform thresholding. The solution using weighted thresholding instead has a low convergence rate due to weighted thresholding strategy. In addition, the resulting grid requires significantly fewer cells using weighted thresholding than using uniform thresholding making it more efficient at approximating the stochastic moments than performing the simulation on an adaptive grid with uniform thresholding.

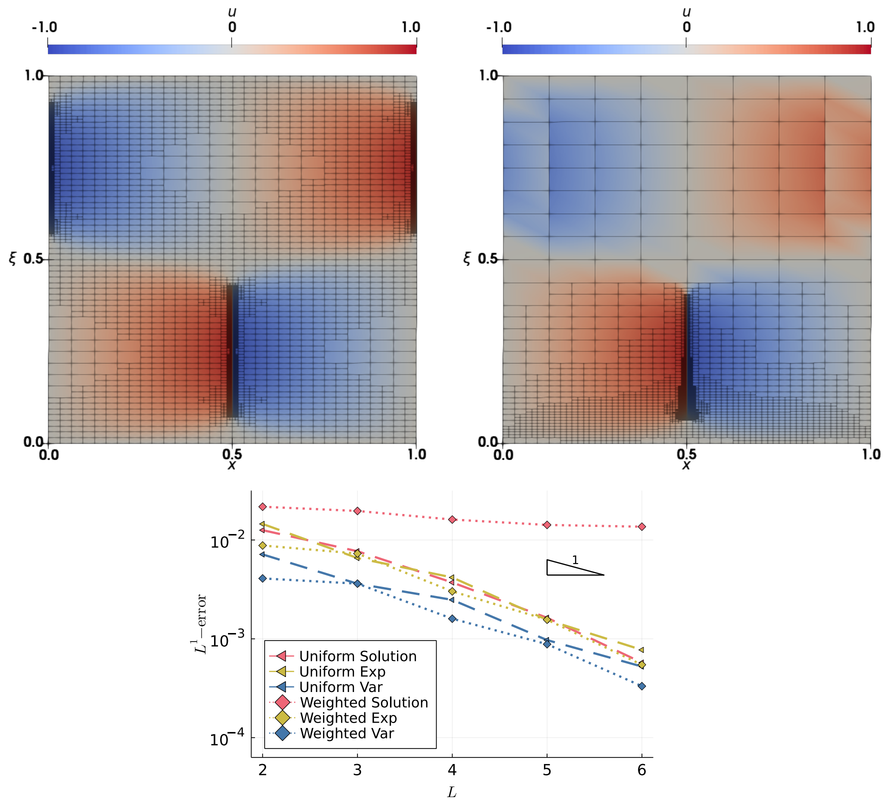

Figure 1: Top Row: Solution for Burgers' equation with smooth uncertain initial data at time $T = 0.35$. Left: Solution and adaptive grid with uniform thresholding; Right: Solution and adaptive grid using weighted thresholding for the random variable $\mathcal{B}(2,20)$; Bottom row: L^1-error of the expectation and variance using uniform and weighted thresholding, respectively.

References

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