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Non Homogeneous Coercive Conservation Laws

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The standard reference for the non homogeneous scalar, 1D, conservation law

$$\partial_t u + \partial_x H(x,u) = 0 \tag{1}$$

is the classical Kružkov paper [4], based on the definition of entropy solutions, on vanishing viscosity approximations and on a precise class of flows H. In particular, a rather restrictive growth condition on H, namely

$$\inf_{\mathbb{R}^2} \partial^2_{xu} H(x,u) > -\infty\,, \tag{2}$$

plays a key role in [4] to obtain those $\mathbf{L}^\infty$ and $\mathbf{BV}$ bounds that are essential in the vanishing viscosity limit.

We present a well posedness result for (1) *alternative* to Kružkov's, relaxing the growth assumption (2) to a mere coercivity condition and slightly weakening the definition of entropy solution. More precisely, on H we require

Smoothness :	$H \in \mathbf{C}^3(\mathbb{R}^2;\mathbb{R})$.
Compact NonHomogeneity :	$\exists\, X > 0 \colon \forall\, (x,u) \in \mathbb{R}^2$ if $\lvert x\rvert > X$ then $\partial_x H(x,u) = 0$;
Uniform Coercivity :	$\forall\, h \in \mathbb{R} \quad \exists\, \mathcal{U}_h \in \mathbb{R} \colon \forall\,(x,u) \in \mathbb{R}^2$ if $\lvert H(x,u)\rvert \le h$ then $\lvert u\rvert \le \mathcal{U}_h$.
Weak Genuine Nonlinearity :	$\forall\, x \in \mathbb{R}$ the set $\left\{w \in \mathbb{R} \colon \partial^2_{ww} H(x,w) = 0\right\}$ has empty interior.

In a more regular setting, the **Uniform Coercivity** assumption would quickly give an $\mathbf{L}^\infty$ bound on solutions by means of characteristics, since they solve an Hamiltonian ODE and lie in level curves of H.

Here, $\mathbf{L}^\infty$ bounds are obtained exhibiting a sufficiently large class of stationary (entropy) solutions to the hyperbolic equation (1) that are not inherited by viscous approximations. More precisely, a first set of $\mathbf{L}^\infty$ bounds applies uniformly to vanishing viscosity approximations whenever the initial datum is more regular, i.e., in $\mathbf{W}^{1,\infty}$. Then, we construct a family, almost a "foliation", of $\mathbf{L}^\infty$ stationary solutions to the hyperbolic equation (1), that allows to replace the well known *Maximum Principle* in [4] when passing to the vanishing viscosity limit for general $\mathbf{L}^\infty$ data. In this construction, discontinuities need to be encompassed and we carefully define shocks so that they turn out to be entropic.

A particular role is that played by the latter assumption above, namely the **Weak Genuine Nonlinearity**. Indeed, it appears unavoidable when passing from a weak to a strong limit, a necessary step that, together with compensated compactness, allows to show that the vanishing viscosity limit is an entropy solution to (1).

Concerning the definition of entropy solution, we have to weaken it avoiding the *"trace at* $0+$ *condition"* [4, Formula (2.2)], due to our making use of compensated compactness arguments. Thus, particular care has to be taken in selecting specific representatives of solutions when defining the semigroup generated by (1). At this point, the results in [5] are of help.

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The connection between (1) and the Hamilton–Jacobi equation

$$\partial_t U + H(x, \partial_x U) = 0 \tag{3}$$

is instrumental to overcome various technical difficulties. Thus, as a byproduct, we detail the correspondence between the semigroup S^{CL} generated by (1) and the semigroup S^{HJ} generated by (3), according to the diagrams

$$\begin{array}{ccc} U_o & \longrightarrow & S_t^{HJ} U_o \\ \partial_x \downarrow & & \downarrow \; \partial_x \\ u_o & \longrightarrow & S_t^{CL} u_o \end{array} \qquad \begin{array}{ccc} U_o & \longrightarrow & S_t^{HJ} U_o \\ \int^x \uparrow & & \uparrow \; \text{Formula (5)} \\ u_o & \longrightarrow & S_t^{CL} u_o \end{array} \tag{4}$$

where

$$\left(S_t^{HJ} U_o\right)(x) = \int_{x_o}^{x} (S_t^{CL} u)(\xi) d\xi - \int_0^t H\left(x_o, (S_\tau^{CL} u)(x_o)\right) d\tau + U_o(x_o)\,. \tag{5}$$

A well posedness result for Hamilton–Jacobi equation (3) is also obtained and, in this connection, the above assumptions are to be compared with those, for instance, in [1, 2, 3]. In this respect, we stress that no convexity assumption is required, not even the semiconcavity [1, Formula (5.36) in Theorem 5.3.9]. The equivalence between the semigroups S^{CL} and S^{HJ} is obtained first at the level of their viscous approximations. The assumption of **Weak Genuine Nonlinearity**, while being of no use in the wellposednes of (3), then appears to be again necessary to rigorously pass the connection (4) at the vanishing viscosity limit.

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