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Axisymmetric swirling flows

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We consider an infinite vortex line in a fluid which interacts with a boundary surface. Motivated by the previous work done in this direction [1], [2], we investigate the types of motion which are compatible with this structure when viscosity is zero. Therefore, we consider the incompressible axisymmetric Euler equations in cylindrical coordinates (r, θ, z) and seek a special class of steady solutions which are exact,

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p, \tag{1a}$$

$$\nabla \cdot \vec{u} = 0, \tag{1b}$$

where $\vec{u}(r,\theta,z) = u(r,z)\vec{e}_r + v(r,z)\vec{e}_\theta + w(r,z)\vec{e}_z$ is the velocity vector and $p(r,\theta,z) = p(r,z)$ is the pressure. In the presence of physical boundaries, the above system is equipped with no-penetration condition on both axis. Namely, we consider the following conditions

$$w = 0 \quad \text{at} \quad z = 0, \quad \text{and} \quad u = 0 \quad \text{as} \quad r \to 0$$

In order to express the equations governing the problem in a self-similar form, we introduce an ansatz in variable $\xi = \frac{z}{r}$,

$$u(r,z) = \frac{1}{r}U(\xi), \quad v(r,z) = \frac{1}{r}V(\xi), \quad w(r,z) = \frac{1}{r}W(\xi)$$
$$p(r,z) = \frac{1}{r^2}P(\xi), \quad \text{and} \quad \psi(r,z) = r\,\theta(\xi)$$

where ψ is a stream function. Recalling the definition of velocity components in terms of a stream function, $u = -\frac{1}{r}\frac{\partial \psi}{\partial z}$ and $w = \frac{1}{r}\frac{\partial \psi}{\partial r}$, we rewrite system (1) as a system of second order differential equations

$$\left[\frac{\theta^2}{2} + (1+\xi^2)P\right]' = -\xi V^2 \tag{2a}$$

$$V'\theta = 0 \tag{2b}$$

$$\left[\theta^2 - \xi\left(\frac{\theta^2}{2}\right)' + P\right]' = 0 \tag{2c}$$

$$\theta' = -U \tag{2d}$$

with the boundary conditions $\theta(0) = 0$ and $\theta'(\xi) \to 0$ as $\xi \to \infty$. Equation (2b) implies that either V is constant, and thus $V' \equiv 0$, or $\theta(\sigma) = 0$ for some $0 \le \sigma \le \infty$.

Firstly we examine the case $V = V_\infty$ is constant. Assuming that $\theta(\xi) \neq 0$, the solutions are obtained by integration involving two constants associated with the swirling velocity V_∞ and the atmospheric pressure P_0. Providing that $D = 2P_0 + {V_\infty}^2 \ge 0$,

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solutions of (2) take the form:

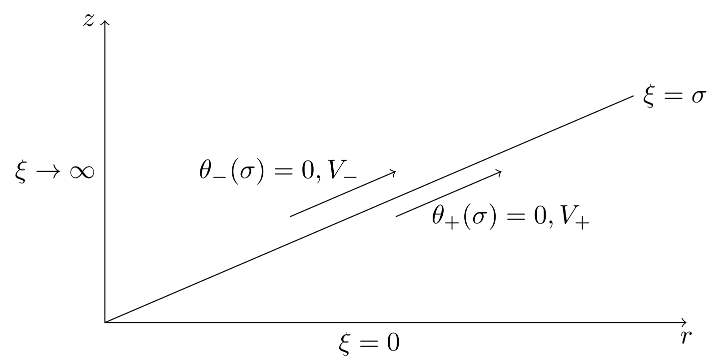
$$\theta = \pm\Big[D\,\xi\big(\sqrt{1+\xi^2}-\xi\big)\Big]^{\frac{1}{2}}$$
$$U = \frac{D}{2\theta}\left[\frac{2\xi\sqrt{1+\xi^2}-(1+2\xi^2)}{\sqrt{1+\xi^2}}\right]$$
$$V = V_\infty$$
$$W = \frac{D}{2\theta}\frac{\xi}{\sqrt{1+\xi^2}}$$
$$P = P_0 - \Big(P_0 + \frac{{V_\infty}^2}{2}\Big)\frac{\xi}{\sqrt{1+\xi^2}}$$

We observe that such flows are either upward or downward along the vortex axis and the direction depends on the sign of θ. This coincides with known results in literature [3].

Let us consider now the case where θ is continuous and there exists $\sigma \in (0,\infty)$ such that $\theta(\sigma)=0,$ [4]. Hence, we seek a class of solutions that satisfy the following jump conditions at $\xi=\sigma$,

$$\Big[P\Big]=0 \quad \text{and} \quad \Big[\theta^2-\xi\theta\theta'\Big]=0$$

This provides an additional restriction on the selection of constants involved in the solution. Solutions in the domain $0 \le \xi \le \sigma$ are now satisfying the boundary conditions $\theta_+(0)=0$ and $\theta_+(\sigma)=0$, while solutions in the domain $\sigma \le \xi \le \infty$ are satisfying the conditions $\theta_-(\sigma)=0$ and $\theta'_-(\xi)\to 0$ as $\xi\to\infty$.



Following the analysis used before, we compute the solutions in each domain and obtain the integration constants $D_- = 2P_- + {V_-}^2$ and $D_+ = 2P_+ + {V_+}^2$ which both need to be positive. In addition, due to the constrain given by the jump condition, we request that

$$\frac{D_+}{D_-} = -\frac{1}{1+2\sigma^2-2\sigma\sqrt{1+\sigma^2}} \tag{4}$$

Clearly, the right hand-side of (4) is negative for all $\sigma\in(0,\infty)$ and this leads to a contradiction. Therefore, we conclude that there is no possible selection of constants D_+ and D_- so that the jump conditions are fulfilled.

References

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