Decomposition of the Wave Manifold into Lax Admissible Regions and its Application to the Solution of Riemann Problems

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We consider the system of two partial differential equations

(1) $$W_t + F(W)_x = 0, \quad \text{with initial conditions} \quad W(x,t=0) = \begin{cases} W_L & \text{if } x < 0, \\ W_R & \text{if } x > 0. \end{cases}$$

We will take $W = (u,v) \in \mathbb{R}^2$ and $F = (v^2/2 + (b_1+1)u^2/2 + a_1 u + a_2 v, uv + a_3 u + a_4 v)^T$ a map $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, the *flux functions*. Equation (1.a) appears in fluid dynamics and together with initial conditions (1.b) is called a Riemann problem.

We are mainly interested in the so called shock solutions, defined by W for $x < st$ and W' for $x > st$, where s is the shock propagation speed, $W = (u,v)$ and $W' = (u',v')$ are the states to be connected by the shock. To have physical meaning as a shock, the speed and the states must satisfy the so called *Rankine-Hugoniot* condition

(2) $$F(W) - F(W') = s(W - W'),$$

for some s. This leads to the definition of Hugoniot curve arc associated to a given state $W = (u_0, v_0)$.

Furthermore, not all arcs of Hugoniot curve arcs are useful to construct solutions. Shock curve arcs must satisfy some extra conditions called admissibility conditions. In [1], Liu's admissibility entropy criterion was introduced within the wave manifold context. There it was shown that under certain extra assumptions, which will be presented in this talk, Liu's entropy criterion can be replaced by the Lax's entropy inequality conditions . These inequalities relate s and the eigenvalues of $DF(u,v)$. Hugoniot curve arcs satisfying these inequalities are called *shock curve arcs or admissible arcs*.

To study Hugoniot curve arcs, we adopt here the topological point of view, as described in [3]. We consider the space $\mathbb{R}^5 = \{(u,v,u',v',s)\}$ and in it, the three-dimensional manifold defined by $F(W) - F(W') = s(W - W')$. Since this manifold is singular along the diagonal $W = W'$, along this diagonal, we perform a blow up, which, in this simple case, is obtained using the coordinate transformations given by $U = (u+u')/2$, $V = (v+v')/2$, $X = u - u'$, $Y = v - v'$, and $Z = Y/X$ and factoring X^2. We also set $c = a_3 - a_2 > 0$. Using these new coordinates, we get

$$(1 - Z^2)(V + a_2) - Z(b_1 U + a_1 - a_4) + c = 0 \quad \text{and} \quad Y = ZX.$$

The two above equations define a regular three-dimensional manifold, which we will call wave manifold and denote by $\mathcal{W}$. In this manifold, we consider the curves defined by (u,v) constant, called Hugoniot curve arcs, their projections onto the phase space are the classical Rankine-Hugoniot curve arcs, see [1]. Since Z is a direction, we may think of the wave manifold $\mathcal{W}$, as contained in $\mathbb{R}^4 \times \mathbb{R}P^1$, or, which is the same, $\mathbb{R}^4 \times S^1$. These coordinates are not valid at $Z = \infty$, but there are no special features at infinity. Here, we perform a new change of variable for the wave manifold $\mathcal{W}$ in the variables (z,t,Y). In these coordinates, we exhibit the main structures obtained through the system of equations.

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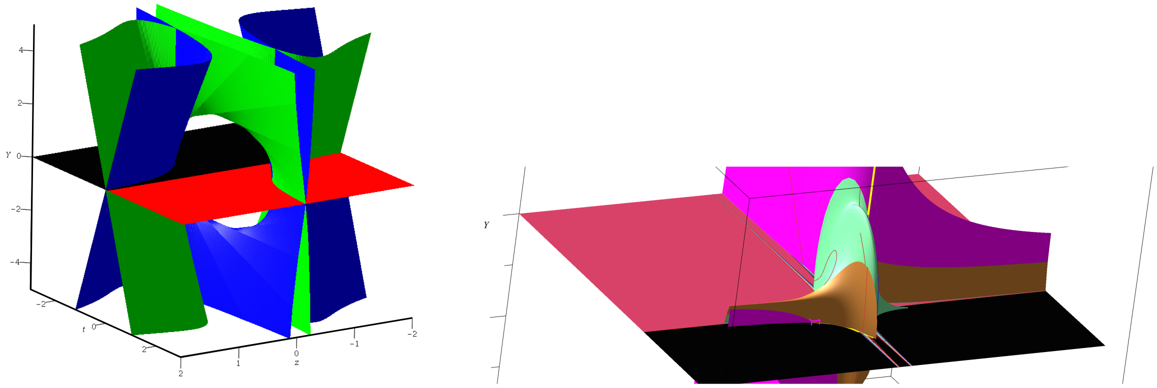

Figure 1: *a: Left).* Decomposition of the wave manifold $\mathcal{W}$ in twelve regions defined by $\mathcal{C} = \mathcal{C}_s \cup \mathcal{C}_f$ (here $\mathcal{C}_s$ is the black half-plane and $\mathcal{C}_f$ is the red half-plane), Son(green surface) and Son'(blue surface). *b: Right).* An example of Riemann solution in the wave manifold. Here we draw the wave curve and the saturated surfaces of this wave curve.

In a series of papers, (see [1, 2, 5, 6]), this manifold and its Hugoniot curve arcs have been studied in the case where F is a polynomial of degree two. In our new system of variables, we define and characterize the relevant surfaces: $Characteristic$ (denoted as $\mathcal{C}$), $Sonic$ (denoted as Son) and $Sonic'$ (denoted as Son'). We also study the intersection of Hugoniot curve arcs with these surfaces and we interpret the Lax's inequalities in this context. Here, also considering F as a polynomial of degree two, we decompose the $characteristic$ and $Sonic'$ surfaces in their fast and slow components and also decompose the wave manifold in regions which we call admissible or non-admissible regions, see Figure 1.a. We restrict ourselves to the symmetric Case IV of the Sheaffer-Shearer classification, see [7]. A study of cases I, II, III and IV (non symmetric case) is found in [4].

In this talk, we review some basic facts and definitions, introduce new variables and describe the $\mathcal{C}$, Son and Son' surfaces, [3]. We characterize the slow $(\mathcal{C}_s)$ and fast $(\mathcal{C}_f)$ components of the characteristic surface associated with the eigenvalues of DF. We also characterize the $coincidence$ curve, which is the boundary of these two components. We describe how the $\mathcal{C}$, Son and Son' surfaces divide the wave manifold into twelve regions, see Figure 1.a, and characterize the surface formed by the Hugoniot curve arcs through points of the $Coincidence$ curve. This surface is tangent to the $Characteristic$ and Son' surfaces. We characterize the slow and fast components of the Son' surface, associated with the slow shock speed and the fast shock speed. We also identify some of regions in the wave manifold where Lax's inequalities are satisfied, indicating in which regions there are local shock curve arcs and in which of these regions there are nonlocal shock curve arcs. We construct also the rarefaction and composite curves in $\mathcal{C}$ and decompose the state space ((u, v)–plane) in elliptic and hyperbolic regions. We utilize before mentioned structures to construct the wave curves from the $\mathcal{C}_s$ plane and we define the $Saturated$ surfaces. We also define the backward two wave and utilize these waves to obtain the Riemann solutions. We present several Riemann solutions in the wave manifold $\mathcal{W}$, see Fig 1.b, and their correspondent solution in the phase space (u, v) to illustrate the developed theory.

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