XVIII International Conference on Hyperbolic Problems
Theory, Numerics, Applications
Málaga, June 20-24, 2022

# Can we compute useful solutions on coarse grids?

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## 1 Introduction

I use the word useful rather than accurate because extreme accuracy is seldom either necessary or possible for most practical problems. In many cases two to three significant figures are very satisfactory, but achieving even this modest goal often incurs substantial expense. The expense could be reduced by using a coarser grid, but the formal concept of accuracy applies only to fine grids. How do we retain some form of accuracy outside the range of the Taylor or Fourier expansions? What concepts other than formal accuracy might be valuable? In this talk I will focus on scheme properties that I believe to be desirable for Computational Fluid Dynamics. None of them are restricted to one-dimensional flow, to small disturbances or low frequencies, but please note that these are not always properties that I know in detail how to achieve.

## 2 Desirable attributes; how to attempt them

### 2.1 Control the flow of information

**Choose appropriate stencils.** It is clearly essential that a numerical stencil includes all of the exact domain of dependence; this is the Courant condition. It should not however include information from far outside the domain of dependence. Even without analysis, it is apparent that this must lead to numerical diffusion. We should obey the Courant restriction, but only just. However, wide stencils are sometimes chosen deliberately, to make a choice of stencil available.

**Avoid semidiscrete methods.** These enlarge the stencil at each stage. This is why increased accuracy, if achieved by increasing the number of stages, implies reduced stability.

**Use compact data representations** The alternative way to increase accuracy is to increase the degrees of freedom within each element. This is done for example in the Discontinuous Galerkin Method, but some of the advantage is lost by the semidiscrete time-stepping.

**Recognise different modes of propagation.** Hyperbolic systems usually involve different modes by which information propagates. Frequently, some information flows with the material, and the domain of dependence for this information is restricted to one dimension along the streamline. Other modes depend on the problem. In elastodynamics, waves that represent tranverse or longitudinal oscillations propagate with speeds that differ, often by a factor of about two. Although two is not very large, in a three-dimensional unsteady calculation it leads to a factor of 16.

### 2.2 Employ correct physics

**Understand multidimensional behavior.** So-called Godunov-type schemes now account for a preponderance of computation for hyperbolic problems. They stem from the question asked in Godunov's original paper "What would be the solution if the data were piecewise constant?" Although this is an excellent question in one dimension, it is only a fairly good question in higher dimensions, because it introduces behavior that is not usually present. It has nevertheless turned out to be a profitable question, apparently because even a loose identification of the domain of dependence is better than none at all.

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**Do not overly focus on shock waves.** An early anxiety was that Godunov-type methods appear to assume that all shockwaves travel normal to cell boundaries. Much effort has been expended to accomodate oblique shockwaves, and this has become known as "multidimensional upwinding". Generally, the overall gain has been small in comparison with the effort involved and there has been no influence on industrial practice.

**Do not insist on shock resolution.** Early research on shock-capturing methods tended to focus on resolving discontinuities over the fewest grid cells so as to reduce storage. Paradoxically, however, the error associated with oblique shocks becomes more serious when the shocks are better resolved; when they are resolved over just one or two cells, in practice the shocks become "staircases". When this requirement is relaxed, it turns out that captured shocks can reproduce complex behavior such as lambda shocks.

**Allow properly for stagnation regions.** Many hyperbolic problems contain elliptic regions, within which the relevance of Riemann problems is extremely dubious. So-called "all-speed" methods, which introduce terms depending on Mach number, have been devised to cancel out undesirable effects. Although improvement is possible, this does not get to the root of the problem. Most errors in the stagnation region take the form of too much dissipation, leading to excessive heating.

**Recognise error progagation.** An unfortunate consequence of errors in the stagnation region is that they propagate downstream, increasing the temperature in an "artificial entropy layer". The same thing can happen with unrealistic shock structures, where local inconsistencies lead to spurious wake behavior.

**Employ proper numerics.** Godunov pointed out that methods that are higher than first-order accurate introduce spurious features in non-smooth regions, unless they are non-linear. This theorem says nothing about the form that the nonlinearity must take. Practical measures have employed flux limiting or flux-corrected transport.

**Avoid spurious "principles".** In one-dimensional problems (especially in the scalar case), there are numerous useful principles available, such as diminishing total variation. These seldom extend to multiple dimensions, but authors are often tempted to invent "maximum principles" and to devise ways of enforcing them.

**Look for invariance.** Multidimensional physics, except under extreme conditions, can be expressed in terms of relationships between quantities that are invariant under a change of coordinates (divergence, vorticity). Such invariant quantities are not typically employed in calculation, but they are often the correct way to express "physics-based" procedures.

## 3 Presentation

I will discuss and show some examples of these issues. Although I am far from possessing a numerical method that accounts for all of the above considerations, I will briefly present the current status of the Active Flux method. Within that context, I will show multidimensional decompositions of the Euler, elastodynamic, and hyperbolic Navier-Stokes equations, and a new method of limiting that can be applied in the absence of any maximum principle.