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On certain nonlocal PDEs describing weakly nonlinear waves

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Surface waves in a fluid have been the object of study of many different researchers since the XVIIIth century [1]. Although in most cases water is assumed to be approximated as an incompressible, irrotational and inviscid fluid, there are situations where the inclusion of viscosity is required to obtain better accuracy. In other circumstances, the fluid has a large enough viscosity so the inviscid assumption is senseless. In both cases the partial differential equations describing the dynamics are

$$\begin{aligned} u_t + (u\cdot\nabla)u - \nabla\cdot\mathcal{T} &= 0 \quad x\in\Omega(t),\ t\in[0,T],\\ \nabla\cdot u &= 0 \quad x\in\Omega(t),\ t\in[0,T], \end{aligned}$$

where u and $\mathcal{T}$ denote the velocity and stress tensor of the fluid respectively and $\Omega(t)$ is the domain occupied by the fluid.

Depending of the fluids under consideration, the specific form of the stress tensor changes. Probably the most celebrated case is the case of even (or shear) viscosity tensor

$$\mathcal{T}^i_j = -p\delta^i_j + \nu_e\left(\nabla_j u_i + \nabla_i u_j\right).$$

Fluids in this family evolve according to the classical Navier-Stokes equations

$$\begin{aligned} u_t + (u\cdot\nabla)u + \nabla p - \nu_e\Delta u &= 0 \quad x\in\Omega(t),\ t\in[0,T],,\\ \nabla\cdot u &= 0 \quad x\in\Omega(t),\ t\in[0,T]. \end{aligned}$$

Analogously, when the viscosity tensor takes the form

$$\mathcal{T}^i_j = -p\delta^i_j + \nu_o\left(\nabla_i u^\perp_j + \nabla^\perp_i u_j\right),$$

with $a^\perp = (a_2, -a_1)$, the viscosity is called odd (or Hall) viscosity. The corresponding equations take the following form

$$\begin{aligned} u_t + (u\cdot\nabla)u + \nabla p - \nu_o\Delta u^\perp &= 0 \quad x\in\Omega(t),\ t\in[0,T],\\ \nabla\cdot u &= 0 \quad x\in\Omega(t),\ t\in[0,T]. \end{aligned}$$

In this talk we will present the derivation of several unidirectional models that describe surface waves for fluids with viscosity. On the one hand, we will present a model for the case of unidirectional surface waves with even viscosity (see also [4, 5, 3, 6]). The resulting partial differential equation takes the following form

$$2f_t = \mathcal{N}\partial_1 f + 2\alpha\mathcal{N}\partial_1^2 f + \mathcal{N}Hf - \beta\mathcal{N}H\partial_1^2 f + \alpha^2\mathcal{N}\partial_1^3 f + Q(f,\partial_1 f,\partial_1^2 f),$$

where H is the Hilbert transform,

$$\mathcal{N} = \left(1-\alpha^2\partial_1^2\right)^{-1}(1-\alpha\partial_1),$$

$\alpha \geq 0$ are dimensionless quantities akin to the Reynolds number, β is the analog of the Bond number and Q denotes a nonlocal quadratic nonlinearity. We observe that this partial differential equation is a nonlinear and nonlocal equation with diffusive

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character. On the other hand, we will present another model for the case of unidirectional surface waves with odd viscosity (see also [2]). In this case, the partial differential equation under consideration reads

$$f_t = \mathcal{M}\partial_1 f + \mathcal{M}Hf + (\alpha - \beta)\mathcal{M}H\partial_1^2 f + P(f, \partial_1 f, \partial_1^2 f),$$

where now the operator $\mathcal{M}$ is defined as follows

$$\mathcal{M} = \left(2 + \alpha\sqrt{-\partial_1^2}\right)^{-1}.$$

When the odd viscosity is considered, the resulting partial differential equation is a nonlinear and nonlocal equation with dispersive character.

Finally, once the derivation has been achieved, a careful mathematical analysis of the well-posedness is performed.

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