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Existence and stability of finite energy solutions to a quantum MHD system

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In this talk I will consider the quantum magneto-hydrodynamic (QMHD) system in three space dimensions

$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\frac{J\otimes J}{\rho}\right) + \nabla P(\rho) = \rho E + J\wedge B + \frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \\ \operatorname{div} E = \rho, \quad \nabla\wedge E = -\partial_t B \\ \operatorname{div} B = 0, \quad \nabla\wedge B = J + \partial_t E. \end{cases} \tag{1}$$

System (1) describes the dynamics of a positively charged quantum fluid, with charge-current density (ρ, J), interacting with its self-generated electromagnetic field (E, B). It is a prototypical model for a quantum plasma, arising for instance in the description of dense astrophysical objects such as white dwarf stars [8]. The introduction of the quantum term

$$\frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \operatorname{div}\left(\frac{1}{4}\rho\nabla^2\log\rho\right) \tag{2}$$

is motivated by the fact that in quantum fluids the thermal de Broglie wavelength becomes comparable to the typical interatomic distance [6]. We will assume $P(\rho)\sim\rho^\gamma$, $\gamma\in(1,3)$, which in this context describes the quantum statistical pressure coming from the electron degeneracy. For instance, for non-relativistic degenerate electron gases, the quantum statistical pressure in the zero temperature limit is given by $P(\rho)\sim\rho^{\frac{5}{3}}$ [4].

We will restrict our attention to solutions to (1) satisfying

$$\nabla\wedge J + \rho B = 2\nabla\sqrt{\rho}\wedge(\rho^{-1/2}J). \tag{3}$$

The above condition reduce to the irrotationality of the velocity field $v := \rho^{-1}J$ outside the vacuum region $\{\rho = 0\}$, and it is compatible with the occurrence of non-zero, quantized circulations around quantum vortices.

The total energy associated to the QMHD system (1) is given by

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\sqrt{\rho}|^2 + \frac{|J|^2}{\rho} + 2f(\rho) + |E|^2 + |B|^2\,dx, \tag{4}$$

where $f(\rho) = \rho\int_0^\rho P(s)/s^2\,ds$ is the internal energy density, and it is formally conserved along the flow.

Models for a quantum (inviscid) fluid usually admit an underlying order parameter, such as the wave-function associated to a Bose-Einstein condensate [7]. For the QMHD system, the order parameter is a function $\psi\in L^2(\mathbb{R}^3,\mathbb{C})$ which satisfies, together with the electromagnetic potential (ϕ, A), the following nonlinear Maxwell-Schrödinger system

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\Delta_A\psi + \phi\psi + f'(|\psi|^2)\psi \\ -\Delta\phi - \partial_t\operatorname{div} A = |\psi|^2 \\ \Box A + \nabla(\partial_t\phi + \operatorname{div} A) = \operatorname{Im}(\bar{\psi}(\nabla - iA)\psi), \end{cases} \tag{5}$$

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where the principal linear part of the Schrödinger equation is governed by the *magnetic* Laplacian

$$-\Delta_A := -(\nabla - iA)^2 = -\Delta + iA\nabla + i\,\mathrm{div}\,A + |A|^2.$$

More precisely, if (ψ, ϕ, A) is a solution to (5), then by defining

$$\rho := |\psi|^2, \quad J := \mathrm{Im}(\bar{\psi}(\nabla - iA)\psi), \quad E = -\nabla\phi - \partial_t A, \quad B = \nabla \wedge A, \tag{6}$$

we have that (ρ, J, E, B) *formally* is a solution to (1), satisfying the generalized irrotationality condition (3). A relevant problem is to find a suitable regularity framework which allows to *rigorously* justify the connection between the QMHD model and the nonlinear Maxwell-Schrödinger system.

In the simpler case of the QHD system (i.e. the compressible Euler equations with the quantum correction (2) in the momentum equation) the situation is well-understood. Indeed the corresponding wave-function dynamics is governed by a standard NLS equation, which is well-posed in the energy space $H^1(\mathbb{R}^3)$. Exploiting this fact, and using a polar factorization argument, in [1] the authors show the existence and stability of global, weak solutions to the QHD system, for every finite energy initial data which are compatible with the underlying NLS structure.

For the QMHD model (1), in which electromagnetic effects are taken into account, the analysis is more delicate. The main technical difficulty is to derive suitable dispersive estimates for Schrödinger operators with rough, time-dependent magnetic potentials, in which case neither spectral methods nor pseudodifferential calculus work sufficiently fine. In fact, well-posedness in the energy space for system (5) is unknown. In [3] the authors settle the *linear* case using the theory of Bourgain-type spaces adapted to magnetic Laplacian, but it is unclear whether such analysis could be extended to the nonlinear case [5].

I will present a different approach for the study of (1), based on the existence of weak solutions to (5) and a direct manipulation of the corresponding integral formulations, without relying on any regularization-stability argument. The regime of regularity considered is almost optimal, namely we take finite energy initial data $(\rho, J, E, B)(t = 0)$ compatible with the structure (6), assuming that the initial magnetic potential is slightly more regular than finite energy, i.e. $A(t = 0) \in H^{1+}$. Under these assumptions, I will show how to deduce suitable Strichartz estimates with $1/2$-loss of derivatives for the magnetic Schrödinger evolution, which are sufficient to justify all the computations with the weak formulation of (5). As a consequence, one deduces the existence of global, finite energy weak solutions to the QMHD system (1) [2].

In the same regularity setting, I will show how to derive a local smoothing estimate for the nonlinear Maxwell-Schrödinger system (5), which allow to prove a weak stability result in the class of solutions discussed above.

I will also briefly discuss the problem of wave-function reconstruction, i.e. the invertibility of the map $(\psi, A) \mapsto (\rho, J)$ defined by (6), which in turn is related to the analysis of the vacuum set $\mathcal{V} := \{(t,x) \,:\, \rho(t,x) = 0\}$.

References

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