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Energy conservation for the compressible Euler equations with vacuum

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In recent years some substantial effort has been directed towards investigating the relation between energy (or, more generally, entropy) conservation and regularity of weak solutions to a given physical system of equations. Onsager's conjecture states that a weak solution of the (three-dimensional) incompressible Euler system will conserve energy if it is Hölder regular with exponent greater than $1/3$. Otherwise it is possible for solutions to exist where anomalous dissipation of energy occurs. Investigating the possibility of analogous statements for other systems has become a lively direction of research. Sufficient regularity conditions for the energy to be conserved were studied for a number of incompressible and compressible models, including the compressible Euler equations.

One of the major differences between incompressible and compressible fluid dynamics is the possible formation of *vacuum* in the latter case. This means that the density of the fluid may become zero in some region. More precisely, consider the isentropic compressible Euler system

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x p(\rho) &= 0, \\ \partial_t \rho + \operatorname{div}_x(\rho u) &= 0, \end{aligned} \tag{1}$$

where u denotes the velocity and ρ the density of the fluid. It is classically known that conservation laws like (1) may develop singularities (shocks) in finite time, which prohibits the use of a smooth notion of solution. Rather, one works with solutions in the sense of distributions, which may be very rough. Suppose now the density were initially bounded away from zero, $\rho^0 \geq c > 0$. If the solution were smooth, then from the continuity equation $\partial_t \rho + \operatorname{div}_x(\rho u) = 0$ it would easily follow that ρ remains bounded away from zero for all times. More precisely, this requires u to have bounded divergence. However, there seems to be no way to guarantee that the velocity component of a weak solution of (1) has bounded divergence, and thus it can not be excluded that the solution spontaneously develops vacuum in finite time. In fact, to our knowledge it remains an outstanding open question whether this can actually occur for the compressible Euler or even Navier-Stokes equations.

The formation of vacuum constitutes a degeneracy that, in many situations, vastly complicates the mathematical analysis of compressible models. For instance, the compressible Euler equations cease to be strictly hyperbolic in vacuum regions. In the context of the current contribution, densities close to zero invalidate the methods and results from previous works, where, it is a crucial assumption that the nonlinearities depend on the dependent variables in a twice continuously differentiable fashion, in order to treat them like a quadratic expression in the commutator estimates. For the system (1), a typical and physically reasonable pressure law would be the polytropic one, i.e. $p(\rho) = \rho^\gamma$ with $\gamma > 1$. The second derivative, however, is of order $\rho^{\gamma-2}$ and thus blows up at zero, at least if $\gamma < 2$. But the regime $1 < \gamma < 2$ is precisely the relevant one (for instance, a monoatomic gas has $\gamma = 5/3$).

We shall present a number of sufficient conditions to ensure energy conservation even after possible formation of vacuum. Based on results from [1].

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