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An Energy Stable Semi-implicit Scheme for the Euler System Under Diffusive Scaling

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1 Isentropic Euler System with Damping

We consider the following non-dimensional barotropic compressible Euler system with damping, parametrised by the Mach number, and posed for $(t, \boldsymbol{x}) \in (0, T) \times \Omega$:

$$\partial_t \rho^\varepsilon + \frac{1}{\varepsilon} \mathrm{div}(\rho^\varepsilon \boldsymbol{u}^\varepsilon) = 0, \tag{1}$$

$$\partial_t (\rho^\varepsilon \boldsymbol{u}^\varepsilon) + \frac{1}{\varepsilon} \mathbf{div}(\rho^\varepsilon \boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) + \frac{1}{\varepsilon} \boldsymbol{\nabla} p^\varepsilon = -\frac{1}{\varepsilon^2} \rho^\varepsilon \boldsymbol{u}^\varepsilon. \tag{2}$$

Here, $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded open connected subset and the dependent variables $\rho^\varepsilon = \rho^\varepsilon(t, \boldsymbol{x}) > 0$ and $\boldsymbol{u}^\varepsilon = \boldsymbol{u}^\varepsilon(t, \boldsymbol{x}) \in \mathbb{R}^d$ denote the fluid density and the fluid velocity respectively. The pressure p^ε is assumed to follow a barotropic equation of state $p^\varepsilon = p(\rho^\varepsilon) := (\rho^\varepsilon)^\gamma$ with $\gamma > 1$ being the ratio of specific heats. The system (1)-(2) is supplied with initial conditions and impermeability boundary condition on the velocity. To avoid unphysical solutions, the system (1)-(2) is further supplemented by the following entropy inequality (see [1, 4]):

$$\partial_t \eta(\rho^\varepsilon, \boldsymbol{u}^\varepsilon) + \mathrm{div} \psi(\rho^\varepsilon, \boldsymbol{u}^\varepsilon) \leq -\rho^\varepsilon |\boldsymbol{u}^\varepsilon|^2 \leq 0 \tag{3}$$

with the entropy, entropy flux pair (η, ψ) given by

$$\eta(\rho, \boldsymbol{u}) = \frac{1}{2} \rho |u|^2 + \psi_\gamma(\rho), \quad \psi(\rho, \boldsymbol{u}) = \left(\frac{1}{2} \rho |u|^2 + \psi_\gamma(\rho) + p(\rho) \right) \boldsymbol{u}. \tag{4}$$

Here $\psi_\gamma \colon \mathbb{R}_+ \to \mathbb{R}$ denotes the Helmholtz function which in the present context measures the internal energy per unit volume.

2 Diffusive limit

At least formally it can be stated that as the small parameter ε goes to 0, the solution $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ converges to the solution $(\bar{\rho}, \bar{\boldsymbol{u}})$ of the compressible porous media equation:

$$\partial_t \bar{\rho} + \mathrm{div}(\bar{\rho} \bar{\boldsymbol{u}}) = 0, \tag{5}$$

$$\nabla p(\bar{\rho}) = -\bar{\rho} \bar{\boldsymbol{u}}. \tag{6}$$

The asymptotic behaviour of the hyperbolic conservation laws with damping have been rigorously studied using the techniques of relative energy estimates in [4, 5].

3 Marker-And-Cell Discretisation and Semi-implicit Scheme

Numerical entropy estimates for space-semi-discrete scheme proposed by of hyperbolic systems with damping have been studied in [1]. In this work we present the following fully discrete semi-implicit scheme for (1)-(2) under a Marker-And-Cell(MAC)

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domain discretisation $\mathcal{T} = (\mathcal{M}, \mathcal{E})$ of Ω (see [2]):

(7)
$$\frac{1}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \frac{1}{\varepsilon}\frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} F_{\sigma,K}(\rho^{n+1}, \boldsymbol{u}^{n+1}) = 0, \ \ \forall K \in \mathcal{M}$$

(8)
$$\frac{1}{\delta t}(\rho_{D_\sigma}^n u_\sigma^{n+1} - \rho_{D_\sigma}^{n-1} u_\sigma^n) + \frac{1}{\varepsilon}\frac{1}{|D_\sigma|}\sum_{\epsilon\in\bar{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}(\rho^n, \boldsymbol{u}^n) u_{\epsilon,\mathrm{up}}^n + \frac{1}{\varepsilon}(\partial_{\mathcal{E}}^{(i)} p^{n+1})_\sigma = -\frac{1}{\varepsilon^2}\rho_{\sigma,\mathrm{up}}^{n+1} u_\sigma^{n+1}, \ \forall \sigma \in \mathcal{E}_{\mathrm{int}}^{(i)}, \ i = 1, 2, \ldots, d,$$

where $\mathcal{M}$ and $\mathcal{E}$ are respectively collections of primal cells and dual edges of the MAC grid, $\mathcal{E}_{\mathrm{int}}^{(i)}$ denotes the collection of internal edges perpendicular to the i-th standard basis of $\mathbb{R}^d$. The primal cells are denoted by K, dual edges are denoted by σ and for each dual edge σ, the corresponding dual cells are denoted by D_σ. The density components are defined on the primal mesh whereas components of the velocity vectors are defined on the dual mesh. The discrete upwind mass and momentum fluxes are defined as similarly as in [2]. The density component of the source term in the momentum update is approximated by the following upwind choice for each internal edge σ separating primal cells K and L

(9)
$$\rho_{\sigma,\mathrm{up}}^{n+1} = \begin{cases} \rho_K^{n+1} & \text{if } u_{\sigma,K}^{n+1} \geq 0, \\ \rho_L^{n+1} & \text{if } u_{\sigma,K}^{n+1} < 0 \end{cases}$$

where $u_{\sigma,K}^{n+1} = u_\sigma^{n+1} \boldsymbol{\nu}_{\sigma,K} \cdot \boldsymbol{e}^{(i)}$ with $\boldsymbol{\nu}_{\sigma,K}$ denoting the outward normal on σ with respect to K and $\boldsymbol{e}^{(i)}$ denoting the i-th standard basis. For further details regarding the discretisation we refer to [2, 3].

The discrete solutions of the proposed numerical scheme satisfies a discrete version of the entropy inequality (3) and hence yields the conditional stability of the scheme which is elaborated in the following lemma:

Proposition 1 *Suppose that for each* $K \in \mathcal{M}$ *and* $\sigma \in \mathcal{E}_{\mathrm{int}}^{(i)}$, $i = 1, 2, \ldots, 3$ $(\rho_K^{n+1}, u_\sigma^n)$ *solves the semi-implicit scheme* (7), (8)*. Also, assume that the time step* δt *satisfies the following* CFL *condition:*

(10)
$$\delta t \leq \min_{i=1,2,\ldots,d} \min_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}} \left\{ \frac{\rho_{D_\sigma}^n |D_\sigma|}{\sum_{\epsilon\in\bar{\mathcal{E}}(D_\sigma)} (F_{\epsilon,\sigma}(\rho^n, \boldsymbol{u}^n))^-} \right\}.$$

Then for all $1 \leq n \leq N-1$*, the following holds:*

(11)
$$\frac{1}{2}\sum_{i=1}^{d}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}} \left(\rho_{D_\sigma}^n |u_\sigma^{n+1}|^2 - \rho_{D_\sigma}^{n-1}|u_\sigma^n|^2\right) + \sum_{K\in\mathcal{M}} |K|(\psi_\gamma(\rho_K^{n+1}) - \psi_\gamma(\rho_K^n)) \leq 0.$$

Formally, the diffusive limit of proposed scheme gives a semi-implicit scheme for the porous media equation with approximate solutions of (5)-(6). Several numerical experiments will be presented to justify the asymptotic behaviour of the scheme.

References

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