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Global weak solutions and blow-up for the Serre–Green–Naghdi equations with surface tension

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The Euler equations are usually used to describe water waves in oceans and Channels. Due to the difficulties to resolve the Euler equations numerically and analytically, several simpler approximations have been proposed in the literature for different regimes. In the shallow water regime, the main assumption is that the mean water depth is small compared to the wave-length. Considering a two-dimensional coordinate system Oxy (Figure 1) and an incompressible fluid layer. Considering the still fluid

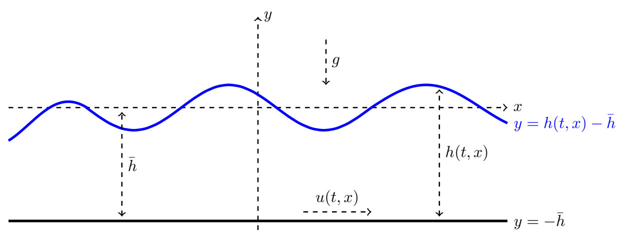

Figure 1: Fluid domain.

level at $y = 0$, the fluid layer is bounded between the flat bottom at $y = -\bar{h}$ and a free surface $y = h(t,x) - \bar{h}$, where h is the total water depth. Assuming long waves in shallow water with possibly large-amplitude, the Serre–Green–Naghdi system (with surface tension) reads

$$
\begin{gathered}
h_t + [h\,u]_x = 0, \\
[h\,u]_t + \left[h\,u^2 + \tfrac{1}{2}\,g\,h^2 + \mathscr{R}_\gamma\right]_x = 0, \\
\mathscr{R}_\gamma := \tfrac{1}{3}\,h^3\left(-u_{tx} - u\,u_{xx} + u_x^2\right) - \gamma\left(h\,h_{xx} - \tfrac{1}{2}\,h_x^2\right),
\end{gathered}
$$

where u denotes the depth-averaged horizontal velocity, g is the gravitational acceleration and $\gamma > 0$ is a constant (the ratio of the surface tension coefficient to the density).

The classical Serre–Green–Naghdi (SGN) system is recovred taking $\gamma = 0$. Smooth solutions of the SGN equations exist locally in time (see [4, 6] for the case $\inf_x h > 0$ and [5] for the shoreline case $\mathrm{sign}(h) = \mathbb{1}_{x>x_0}$). The questions of the blow-up of smooth solutions and the existence of global solutions of the classical SGN equations are still open.

We consider in this talk the Serre–Green–Naghdi equations with surface tension (γ-SGN). This system have been derived in [1] as a generalisation of the classical SGN equations ($\gamma = 0$). Due to the appearance of time derivatives in the definition of $\mathscr{R}$, it is convenient to apply the inverse of the Sturm–Liouville operator $\mathcal{L}_h := h - \frac{1}{3}\,\partial_x\,h^3\,\partial_x$, the γ-SGN system becomes then

$$
h_t + [h\,u]_x = 0, \tag{1a}
$$

$$
u_t + u\,u_x + g\,h_x = -\mathcal{L}_h^{-1}\partial_x\left\{\tfrac{2}{3}\,h^3\,u_x^2 - \left[\gamma\,h - \tfrac{1}{3}\,g\,h^3\right]h_{xx} + \tfrac{1}{2}\,\gamma\,h_x^2\right\}. \tag{1b}
$$

The local well-posedness of (1) have been established in [4]. Smooth solutions of (1) satisfy the energy conservation $\mathscr{E}_t + \mathscr{D}_x = 0$, where

$$
\mathscr{E} := \tfrac{1}{2}h\,u^2 + \tfrac{1}{2}\,g\,(h - \bar{h})^2 + \tfrac{1}{6}\,h^3\,u_x^2 + \tfrac{1}{2}\,\gamma\,h_x^2.
$$

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We present in this talk a precise blow-up criterion of (1) obtained in [2] and we build a class of small-energy initial data such that such scenario occurs. We also show the existence of a strongly continuous semigroup of global weak dissipative solutions for any small-energy initial data [3]. The Riemann invariants of the weak solutions satisfy the one-sided Oleinik inequality

$$u_x \pm \sqrt{3\gamma}\, h^{-\frac{3}{2}}\, h_x \leqslant C\left(1+t^{-1}\right).$$

References

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