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Traffic flow models with nonlocal velocity: The singular limit problem

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The progress in autonomous driving brings new challenges for the modeling of traffic flow. Therefore, very recently, so-called nonlocal traffic flow models have been introduced [1, 2, 5, 8]. They include more information in a certain nonlocal range about the traffic of the road. In case of non-autonomous drivers, the nonlocal range can stand for the sight of a human driver. Moreover, this nonlocal range can be interpreted as the connection radius of autonomous cars. As at the moment not every car is connected or even autonomous, nonlocal traffic flow models work with a limited interaction range.

Here, we will present a nonlocal traffic flow model in which drivers adapt their speed based on a mean downstream velocity, anticipating the future space in front of them. This leads to a slightly different model than the usual one being based on a mean downstream density. The model is given by

$$\text{(NV)} \qquad \partial_t \rho + \partial_x \left(\rho \left(W_\eta * v(\rho)\right)\right) = 0, \quad \text{where} \quad \left(W_\eta * v(\rho)\right)(t,x) := \int_x^{x+\eta} v(\rho(t,y)) W_\eta(y-x)\,dy, \quad \eta > 0,$$

with

$$\text{(1a)} \qquad v \in C^2(I;\mathbb{R}^+) \text{ with } v' \le 0,$$

$$\text{(1b)} \qquad W_\eta \in C^1([0,\eta];\mathbb{R}^+) \text{ with } W_\eta' \le 0, \ \int_0^\eta W_\eta(x)\,dx = 1 \ \forall \eta > 0.$$

and the dynamics are coupled to the initial conditions

$$\text{(2)} \qquad \rho(0,x) = \rho_0(x) \in L^1 \cap \mathrm{BV}(\mathbb{R}; I), \quad I = [0, \rho^{\max}] \subseteq \mathbb{R}^+,$$

for some $\rho^{\max} > 0$. The advantage of such an approach is that changes in the speed law can be naturally included, see e.g. [4]. We are able to give the following well-posedness result:

THEOREM 1 *Under the Assumptions* (1) *and* (2)*, the Cauchy problem given by* (NV) *and* (2) *admits a unique weak solution for any $T > 0$ with*

$$\inf_{\mathbb{R}}\{\rho_0\} \le \rho(t,x) \le \sup_{\mathbb{R}}\{\rho_0\} \quad \textit{for a.e. } x \in \mathbb{R},\ t \in (0,T].$$

We will shortly talk about the existence of weak solutions, as proven in [8], and their uniqueness, which can be similarly obtained as in, e.g. [9].

When considering the model (NV) with a fixed range $\eta > 0$, there are two natural questions which arise concerning the limits of this range: What happens if the interaction range tends to zero or, conversely, what happens if it tends to infinity (so every car is connected)?

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Concerning the latter question, we can give the following answer, see also [7]:

PROPOSITION 2 *Let the Assumptions* (1) *and* (2) *hold and in particular, let* ρ_0 *be of compact support. Assume additionally*

$$\lim_{\eta\to\infty} W_\eta(0) = 0.$$

Then, the solution ρ^η *of* (NV) *converges for* $\eta \to \infty$ *to the unique entropy solution of the linear transport equation*

$$\partial_t \rho + \partial_x (v(0)\rho) = 0,$$
$$\rho(0,x) = \rho_0(x).$$

So in an idealistic (but unrealistic) setting of an infinite connection range traffic jams would disappear.

Conversely, if $\eta \to 0$, numerical examples suggest a convergence to the local Lighthill-Whitham-Richards model, i.e.,

$$\partial_t \rho + \partial_x (v(\rho)\rho) = 0.$$

Similar to the nonlocal traffic flow model based on the mean downstream density, the BV estimates used to establish existence of our model (NV) blow up for $\eta \to 0$. Nevertheless, we will present some recent results obtained in [6] in which BV bounds can be established such that we answer the first question at least partly:

- **Specific initial data, arbitrary kernel:** In the case of monotone increasing or decreasing initial data we are able to prove that under the assumptions (1a) the monotonicity is preserved and hence together with the maximum principle the total variation is bounded for all kernels satisfying (1b). Then, we are able to show that in the limit the entropy condition of the LWR model is satisfied. Note that in contrast to the model of [2] no further restrictions on the velocity are necessary to preserve the monotonicity, compare [10].
- **Arbitrary initial data, specific nonlocal term:** We consider an exponential kernel and a slightly modified conservation law in comparison to (NV), i.e.

$$\partial_t \rho + \partial_x \left(\rho \left(\frac{1}{\eta} \int_x^\infty \exp\left(\frac{x-y}{\eta} \right) v(\rho(t,y)) dy \right) \right) = 0.$$

Under some additional assumptions on the velocity function, we are able to prove the convergence and entropy admissibility for $\eta \to 0$ and the LWR model using a similar idea as in [3].

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