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On a Hamiltonian regularised Burgers equation

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Nonlinear hyperbolic partial differential equations (the inviscid Burgers, Saint-Venant equations for example) are known to develop discontinuous shocks in finite time, even if the initial data is a C^∞ function. In order to avoid those discontinuities, small diffusion and/or dispersion terms can be added to the equation [1, 7, 6, 8]. The diffusion term leads to a loss of energy even for smooth solutions and the dispersion term can lead to spurious oscillation which are not very practical for simulations for large time. In [5], we proposed the regularised Burgers (rB) equation

(1) $$u_t + u\,u_x = \ell^2\,(u_{txx} + 2\,u_x\,u_{xx} + u\,u_{xxx})\,,$$

where ℓ is a positive parameter. The rB equation is Galilean invariant, non dispersive and conserves the H^1 norm of the solution. This regularisation have been generalised for scalar conservation laws [4]. It should be noted that the rB equation (1) is similar to the well-known dispersionless Camassa–Holm equation [2]

(2) $$u_t + 3\,u\,u_x = \varepsilon\,(u_{txx} + 2\,u_x\,u_{xx} + u\,u_{xxx})\,.$$

However, the Camassa–Holm equation is not Galilean invariant. Due to the appearance of the time derivatives on the right hand-side of (1), it is convenient to apply the operator $(1-\ell^2\partial_x^2)^{-1}$ on (1) to obtain

(3) $$u_t + u\,u_x + \tfrac{1}{2}\ell^2\,\mathfrak{G} * u_x^2 = 0, \qquad \mathfrak{G}(\lambda) := e^{-\frac{|\lambda|}{\ell}}/(2\,\ell).$$

We present in this talk the existence of a semigroup of global weak solutions of (3) established in [3]. We present some properties of the rB equation (3) and we show that the solutions satisfy a uniform (on ℓ) Oleinik inequality. This leads to a uniform bound of the total variation of the solutions [5]. Using those uniform estimates and some compactness arguments we show that as $\ell \to 0$ the weak solution given by the semigroup of (3) converges "up to a subsequence" to u^0 which is the unique entropy solution of the Burgers equation

$$u^0_t + \tfrac{1}{2}\left[(u^0)^2\right]_x = 0, \qquad u^0_x \leqslant 2/t.$$

Also, as $\ell \to \infty$, the weak solution given by the semigroup of (3) converges "up to a subsequence" to u^∞ satisfying the Hunter–Saxton equation

$$[u^\infty_t + u^\infty\,u^\infty_x]_x = \tfrac{1}{2}\,[u^\infty_x]^2\,, \qquad u^\infty_x \leqslant 2/t.$$

References

[1] H. S. Bhat and R. C. Fetecau. A Hamiltonian regularization of the Burgers equation. *J. Nonlinear Sci.*, 16(6):615–638, 2006.

[2] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Physical review letters.*, 71(11):1661, 1993.

[3] G.M. Coclite, H. Holden and K.H. Karlsen. Global weak solutions to a generalized hyperelastic-rod wave equation. *SIAM Journal on Mathematical Analysis*, 37(4):1044–1069, 2005.

[4] B. Guelmame. On a Hamiltonian regularization of scalar conservation laws. *Working draft*, 2022.

[5] B. Guelmame, S. Junca, D. Clamond, and R. Pego. Global weak solutions of a Hamiltonian regularised Burgers equation. *Preprint*, 2020.

[6] B. T. Hayes and P. G. LeFloch. Nonclassical shocks and kinetic relations: strictly hyperbolic systems. *SIAM J. Math. Anal.*, 31(5):941–991, 2000.

[7] P. D. Lax and C. D. Levermore. The small dispersion limit of the kdv equations I. *Comm. Pure Appl. Math.*, 3:253–290, 1983.

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[8] J. VonNeumann and R. D. Richtmyer. A method for the numerical calculation of hydrodynamic shocks. *J. Appl. Phys.*, 21(3):232–237, 1950.