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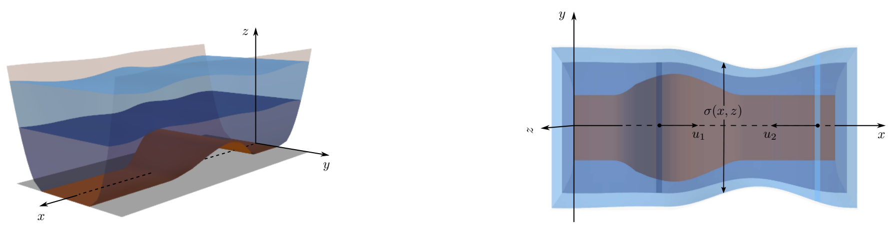
# A central-upwind scheme for two-layer shallow-water flows with friction and entrainment along channels

Gerardo Hernandez-Duenas [*], Jorge Balbas [†]




(a) (b)

Figure 1: Schematic of a two layer channel flow with the two layers moving in opposite directions: (a) Full 3D view of the flow, (b) Top view.

In this work we present a new high-order numerical scheme for simulating two-layer shallow-water flows along channels with a bottom topography and varying width (see Figure 1). These flows are characterized by a large horizontal length scale relative to their depth and are commonly observed in nature –*e.g.*, channel flows, straits, mountain passes; and their modeling and simulation have applications in flood control, coastal engineering, or environmental assessment among others.

The various challenges that the Saint-Venant equations pose for simulating shallow-water flows are well known and have been studied extensively. The two-layer Saint-Venant system is conditionally hyperbolic, and the common occurrence of steady states in geophysical flows require *well-balance* numerical schemes capable of capturing and resolving accurately steady-state solutions of the PDE model. These challenges become increasingly difficult to address as the flows and the corresponding PDE models that describe them increase in complexity, and a significant effort over the past couple of decades have lead to the development of numerical schemes for simulating a wide variety of flows. Numerous schemes have been proposed for the simplest case of one-layer flows along channels with constant width: a positivity preserving *kinetic scheme* capable of preserving the steady state of rest is presented in [12], and in [1] a finite volume scheme with similar properties is devised using *hydrostatic reconstruction* to capture steady-state solutions. The authors of [9] proposed a *discontinuous Galerkin* method. And [10, 4] introduce positivity preserving well-balance *central-upwind* schemes for these flows. The approach suggested to achieve well-balance and positivity within the central-upwind framework is extended in [2] to simulate flows along channels with *varying cross-sections* using a *central* scheme, a type of flows also addressed in [7] using an upwind Roe-type scheme. And the authors of this paper proposed a new central-upwind scheme for one-layer flows along channels with arbitrary geometry in [3]. For flows along channels with arbitrary geometry, the authors of [6] extended with great success (and high impact in the field) the $Q$-scheme for hyperbolic systems with source terms previously introduced in [5].

We propose a new high-order central-upwind scheme to compute two-layer shallow-water flows along channels with arbitrary geometry that incorporates the treatment of friction and entrainment terms. The model in the absence of friction and entrainment can be found in [6]. These new terms allow us to simulate more realistic flows and to assess the limitations of the model and the scheme. In order to understand and address the challenges posed by the model and its limitations, we present a detailed analysis of the hyperbolic PDE model, with special emphasis on the conditions that lead to the loss of hyperbolicity. To this

[*]Institute of Mathematics, National University of Mexico, Blvd. Juriquilla 3001, Queretaro, Mexico. Email: sarswatishah@im.unam.mx
[†]Department of Mathematics. California State University, Northridge . Email: jorge.balbas@csun.edu

end, we derive rigorous asymptotic approximations of the eigenvalues of the quasilinear form of the model and, for the sake of completeness in the analysis, we prove the existence of an entropy function and an entropy inequality that physically relevant weak solutions must satisfy.

In order to address the various numerical and computational challenges we propose a central-upwind scheme based on the semi-discrete central-upwind schemes for hyperbolic conservation laws of Kurganov, Noelle, and Petrova, [11], characterized by their simple implementation and robustness. The proposed numerical scheme evolves the cell averages of the flow variables with second order accuracy, and their implementation requires four main ingredients: a *non-oscillatory reconstruction* of point values from cell averages that preserves the *positivity* of the water depth, an *evolution routine* to advance the solution in time, *estimates* of the largest and smallest eigenvalues of the system, and the *discretization of source terms and non-conservative products* that balance the hyperbolic fluxes so as to recognize steady states at rest and add accuracy to the computation of flows near non stationary steady states.

Besides showing details of the mathematical properties of the system an analysis of the hyperbolicity of the system and eigenvalue bounds, numerical tests and possible applications will be included. See [8] for more details.

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