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Asymptotic-preserving methods for hyperbolic problems with stiff source terms

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Some fluid and kinetic systems of equations in the presence of (potentially multiple) small parameters admit so-called asymptotic regimes, where they reduce to a smaller set of equations, potentially with a different mathematical structure. However, classic numerical approaches, such as the usual finite volume methods, do not naturally degenerate in these asymptotic regimes to consistent discretizations of the limit equations. Furthermore, even though stability conditions usually become more and more restrictive when we approach these asymptotic regimes, meaning smaller and smaller time steps, accuracy can be dramatically reduced and the results frequently unexploitable. Asymptotic-preserving schemes are designed to both lift the restrictive stability conditions and remain accurate in the asymptotic regime.

In this work, the isentropic Euler equations with friction stands as our prototype nonlinear hyperbolic problem with stiff relaxation. Using the scaling that can be found for instance in [1] this model reads:

$$\partial_t \rho + \frac{1}{\varepsilon}\partial_x(\rho u) = 0, \tag{1a}$$

$$\partial_t(\rho u) + \frac{1}{\varepsilon}\partial_x\left(\rho u^2 + p(\rho)\right) = -\frac{\sigma}{\varepsilon^2}\rho u, \tag{1b}$$

where ρ is the density of the fluid and u its macroscopic velocity, σ is the collision rate and ε is the small parameter that characterizes the asymptotic regimes. Here we only assume that the pressure term verifies $p'(\rho) > 0$. This system of equations (1) reduces (see for instance [1]) to a parabolic equation in the limit of long-term dynamics and high friction, that is when $\varepsilon \to 0$:

$$\partial_t \rho - \partial_x\left(\frac{1}{\sigma}\partial_x\left(p\left(\rho\right)\right)\right) = 0. \tag{2}$$

Existing asymptotic-preserving approaches of high order either do not consider Euler equations but simpler nonlinear systems [2, 3] or are not fully of high order with respect to both time and space [1, 4]. The new formalism that we introduce allows to design an asymptotic-preserving second-order in time and space method for the Euler-friction equations. The method uses the Implicit-Explicit (IMEX) framework (see for instance [2, 3]) to make the scheme partially implicit, both with regard to the source term and to the flux, without having to solve a linear system. This allows to significantly alleviate stability constraints as well as to retrieve uniform accuracy in all asymptotic regimes. The accuracy and stability properties have been theoretically studied on a linearized version of the equations and numerically validated for the full nonlinear system. Our aim also includes plasma discharges with sheaths [5], where we have two small parameters related to Debye length and mass ratio. Such applications are modeled using a two-fluid approach (ions+electrons) with Euler system coupled to the Poisson equation. Numerical simulations assess and illustrate the potential of the method we have introduced.

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