XVIII International Conference on Hyperbolic Problems
Theory, Numerics, Applications
Málaga, June 20-24, 2022

Well-balanced implicit-explicit Lagrange-projection scheme for two-layer shallow water equations

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We are interested in the numerical approximation of the 1D two-layer shallow water system, with which we assume the fluid to be composed of two superimposed layers of immiscible liquids where the upper one has a smaller density ρ. Thus, using the subscript $j = 1, 2$ to indicate the jth layer, we state $\rho_1 < \rho_2$. This kind of situations can occur when there are two liquids of different densities or even with a single fluid present at two different temperatures, as in oceanic flows. Referring for instance to [2] and also to figure 1 for the notations, the non-conservative two-layer shallow water system is given by

$$\begin{cases} \partial_t h_1 + \partial_x(h_1 u_1) = 0 \\ \partial_t(h_1 u_1) + \partial_x(h_1 u_1^2 + \dfrac{g h_1^2}{2}) + g h_1 \partial_x h_2 = -g h_1 \partial_x z \\ \partial_t h_2 + \partial_x(h_2 u_2) = 0 \\ \partial_t(h_2 u_2) + \partial_x(h_2 u_2^2 + \dfrac{g h_2^2}{2}) + g \dfrac{\rho_1}{\rho_2} h_2 \partial_x h_1 = -g h_2 \partial_x z \end{cases} \tag{1}$$

where $t > 0$ represents the time and x the axial coordinate. Then, $h_j(x,t) > 0$ is the water depth of the corresponding layer, $u_j(x,t)$ the averaged velocity and finally $z(x,t)$ the bed elevation. Finally, we define the gravitational acceleration g and $r = \frac{\rho_1}{\rho_2}$ the relative density. Regarding the latter, depending on its value, we may be able to explicitly define the eigenvalues of system (1) or not. In particular, we are interested in situations in which $r \approx 1$, which often happen in geophysical flows. In this case, first-order approximation of the eigenvalues are available. However, an additional difficulty is that the system is only conditionally hyperbolic, namely complex internal eigenvalues could appear. From a physical point of view, this could resemble the mixing of the two layers, itself due to the appearance of shear instabilities.

Concerning the numerical strategy, here we aim to design and implement well-balanced implicit-explicit Lagrange-projection schemes. So far Lagrange-Projection (LP) methods have been studied for different mathematical models as the shallow water system [3] and the gas dynamic equations [4], among the others. However, up to our knowledge, they have never been employed to numerically approximate the two-layer shallow water equations. Indeed, due the presence of two velocities u_1, u_2, it is not straightforward to understand how to apply the Lagrange-projection strategy to this system. A first idea could be to implement the LP approach for each layer and then to couple them. However, it is known that a method that applies an arbitrary scheme to each layer usually leads to the presence of spurious oscillations in the numerical results [2]. Moreover, in [5] the authors described a

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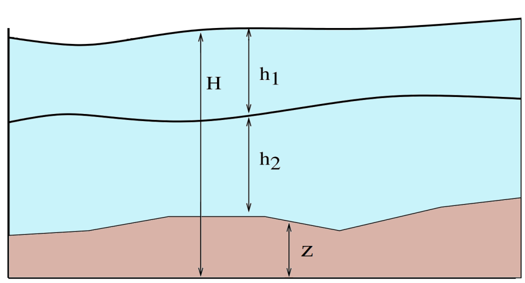

Figure 1: Sketch of the two-layer shallow water: h_1, h_2 water heights, z topography and H free surface.

first attempt to apply the Lagrange-projection strategy to a two-phase system, in particular the two-fluid two-pressure (or seven-equation) model. There, the coupling terms of the system have not been considered directly inside the Lagrange-projection decomposition but in a third step. Here we propose a coupled approach, different from the one mentioned above.

Furthermore, we also consider a different interpretation of the Lagrange-projection approach, namely the acoustic-transport splitting, refer again to [3, 4]. Indeed, by decomposing the different phenomena of the mathematical model, we obtain two different systems, the acoustic and transport one. For the former, we design an approximate Riemann solver based on a relaxation approach and then the associated Godunov-type scheme is used. We also explain how the resulting approximation can be exploited for the Lagrangian system. Furthermore, let us recall that the acoustic-transport splitting (or equivalently the Lagrange-projection decomposition) can be particularly interesting in subsonic regimes, where the acoustic waves are much faster than the transport ones. This means that an implicit approximation applied to the acoustic system could lead to the construction of very fast numerical schemes as we would neglect the acoustic time step condition. For this reason, we propose both an explicit and an implicit strategy for the acoustic equations, while keeping an explicit approximation for the transport step. For implicit-explicit LP methods refer for instance to [3].

Last but not least, we are interested in the well-balanced property of the numerical schemes, meaning that the numerical methods are able to preserve the stationary solutions of the mathematical model. Indeed, it is well-known that otherwise we could observe unphysical oscillations in the numerical simulations when near to a steady state. Refer for instance to [7] and to [1] for well-balanced schemes with and without the Lagrange-projection decomposition respectively. Here we are particularly interested in preserving only the stationary solutions with zero velocity, namely

$$\begin{cases} u_j = 0, \quad \text{with} \quad j = 1, 2 \\ h_1 = \text{constant} \\ h_2 + z = \text{constant}. \end{cases} \tag{2}$$

Finally, numerical simulations are carried out in order to validate the numerical methods.

Acknowledgements

This is a joint work with M. J. Castro Dìaz, C. Chalons and T. Morales de Luna.

This research has been partially supported by a grant from Région Île-de-France.

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