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Second order finite volume IMEX-RK numerical methods for 1d models in option pricing

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This work deals with the development of second order finite volume numerical schemes for solving option pricing problems, modelled by low dimensional advection-diffusion-reaction scalar partial differential equations. These equations will be discretized using second order finite volume Implicit-Explicit (IMEX) Runge-Kutta (RK) schemes. The developed methods will be able to overcome the time step restriction due to the strict stability condition of parabolic problems with diffusion terms. Besides, the schemes will offer high-accurate and non oscillatory approximations of option prices and their Greeks.

In this work we consider the Black-Scholes (BS) option pricing model, where the evolution of the price of the risky asset is given by the Stochastic Differential Equation (SDE) $\frac{ds_t}{s_t} = (r-q)dt + \sigma dW_t$, with W_t a standard Brownian motion. The parameter $r \in \mathbb{R}$ is the risk free constant interest rate and $q \in \mathbb{R}$ is the continuous dividend yield. The parameter $\sigma \in \mathbb{R}^+$ is the volatility of the stock price, which is again considered as constant. The BS model is based on several assumptions, like for example the fact that the volatility of the underlying asset is a deterministic constant. Although nowadays all of these assumptions about the market can be shown wrong up to a certain extent, the BS model is still very important in theory and practice, and it has a huge impact on financial markets. The price u of any option on the underlying s is fully determined at every instant t by the asset value s_t. Hence, the value of the option is a function $u(s,t)$. Applying Itô's lemma, one can derive the SDE for u. In order to comply with the no-arbitrage conditions, the process du has to be martingale. Therefore, the drift term of the SDE for u must be zero, which implies the well-known linear parabolic backward in time Black-Scholes PDE

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} - (r-q)s\frac{\partial u}{\partial s} + ru = 0, \quad (s,t) \in [0,\infty) \times [0,T]. \tag{1}$$

The initial condition $u(s,0)$ depends on the payoff of the option and the boundary conditions should be carefully determined taking into account financial aspects as well as mathematical questions. In this work we study several types of options will be described, together with their corresponding initial and boundary conditions. In this work we present a second order finite volume semi-implicit numerical scheme for solving (1). First, the equation (1) must be written in conservative form: $\frac{\partial u}{\partial t} + \frac{\partial}{\partial s} f(u) = \frac{\partial}{\partial s} g(u_s) + h(u)$. The numerical solution of this equation using a explicit finite volume scheme may have a huge computational cost because of the tiny time steps induced by the diffusive terms. To avoid this difficulty we consider IMEX Runge-Kutta methods (see [1]).

Spatial semi-discretization: finite volumes. The discretization of advective and source terms is performed by means of a explicit finite volume scheme, see [2, 3]. The spatial domain is split into cells (finite volumes) $\{I_i\}$. The unknowns of our problem are the averages of the solution $u(s,t)$ in the cells $\{I_i\}$, $\{\bar{u}_i(t)\}_i$. Integrating the conservative equation in space on I_i and dividing by $|I_i|$:

$$\frac{d\bar{u}_i}{dt} = -\frac{1}{|I_i|}\left[f(u(s_{i+1/2},t)) - f(u(s_{i-1/2},t))\right] + \frac{1}{|I_i|}\left[g(u_s(s_{i+1/2},t)) - g(u_s(s_{i-1/2},t))\right] + \frac{1}{|I_i|}\int_{I_i} h(u)\,ds. \tag{2}$$

The convective terms in (2) can be approximated by solving the Riemann problems at the edge of the cells using a suitable numerical flux function $\mathcal{F}$ consistent with the analytical flux f, i.e. $f(u(s_{i\pm1/2},t)) \approx \mathcal{F}(u^-_{i\pm1/2}, u^+_{i\pm1/2})$. The quantities $u^\pm_{i\pm1/2}$ are computed as $u^\pm_{i\pm1/2} = \lim_{s\to s^\pm_{i\pm1/2}} \mathcal{R}(s)$, where $\mathcal{R}$ is a reconstruction of the unknown function $u(s,t)$. More precisely, $\mathcal{R}$ is given by a piecewise polynomial built from cell averages $\{\bar{u}_i(t)\}$. The integral of the source term (2) can be explicitly discretized

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using a second order quadrature rule, for example the midpoint rule. Finally, the diffusion term can be approximated implicitly as: $g(u_s(s_{i+1/2})) - g(u_s(s_{i-1/2})) \approx g(\frac{\bar{u}_{i+1}-\bar{u}_i}{|I_i|}) - gt(\frac{\bar{u}_i-\bar{u}_{i-1}}{|I_i|})$.

Time discretization: IMEX-RK. After performing the spatial semi-discretization we obtain a stiff ODE system in time of the form $\frac{\partial U}{\partial t} + F(U) = S(U)$, where $U = (\bar{u}_i(t))$ and $F, S : \mathbb{R}^N \to \mathbb{R}^N$, being F the non-stiff term and S the stiff one. An IMEX scheme consists of applying an implicit discretization to the stiff term and an explicit one to the non stiff term. In this way, both can be solved simultaneously with high order accuracy using the same *time step* of the convective part, which is in general much larger than the time step of the diffusive part. When IMEX is applied to the ODE system it takes the form

$$U^{(k)} = U^n - \Delta t \sum_{l=1}^{k-1} \tilde{a}_{kl} F(t_n + \tilde{c}_l \Delta t, U^{(l)}) + \Delta t \sum_{l=1}^{\rho} a_{kl} S(t_n + c_l \Delta t, U^{(l)}), \tag{3}$$

$$U^{n+1} = U^n - \Delta t \sum_{k=1}^{\rho} \tilde{\omega}_k F(t_n + \tilde{c}_k \Delta t, U^{(k)}) + \Delta t \sum_{k=1}^{\rho} \omega_k S(t_n + c_k \Delta t, U^{(k)}), \tag{4}$$

where $U^n = (\bar{u}_i^n)$ is the vector of the unknowns at times t^n, and $U^{(k)}$ and $U^{(l)}$ are the vector of unknowns at the stages k, l of the IMEX method. The matrices $\tilde{A} = (\tilde{a}_{kl})$, with $\tilde{a}_{kl} = 0$ for $l \geq k$, and $A = (a_{kl})$ are square matrices of order ρ, such that the ensuing scheme is implicit in S and explicit in F. Solving efficiently at each time step the system of equations corresponding to the implicit part is extremely important. In this work we will consider the second order diagonally implicit Runge-Kutta (DIRK) L-stable scheme IMEX-SSP2(2,2,2) (see [1]). An explicit time integrator needs extremely small time steps due to the following stability conditions $\eta \frac{\Delta t}{(\Delta s)^2} \leq \frac{1}{2}, \alpha \frac{\Delta t}{\Delta s} \leq 1$, where $\eta = \left|\frac{\partial g}{\partial u_s}\right|$, $\alpha = \left|\frac{\partial f}{\partial u}\right|$, for all cells I_i and for all boundary points $s_{i\pm 1/2}$. However, IMEX only needs to satisfy the advection stability condition $\alpha \frac{\Delta t}{\Delta s} \leq 1$ which results in high gains in computational time.

	IMEX			
N	L_1 error	Order	Δt	Time (s)
50	1.6145×10^1	$--$	1.01×10^{-1}	2.8×10^{-4}
100	7.1629×10^0	1.17	5.03×10^{-2}	4.7×10^{-4}
200	2.6877×10^0	1.41	2.50×10^{-2}	1.18×10^{-3}
400	9.1734×10^{-1}	1.55	1.25×10^{-2}	3.6×10^{-3}
800	2.8046×10^{-1}	1.70	6.26×10^{-3}	1.1×10^{-2}
1600	7.2788×10^{-2}	1.95	3.13×10^{-3}	2.6×10^{-2}
3200	1.7410×10^{-2}	2.06	1.56×10^{-3}	9.5×10^{-2}
6400	3.4791×10^{-3}	2.32	7.82×10^{-4}	3.5×10^{-1}
	Explicit			
N	L_1 error	Order	Δt	Time (s)
50	1.6146×10^1	$--$	1.01×10^{-1}	1.1×10^{-4}
100	7.1626×10^0	1.17	5.03×10^{-2}	1.9×10^{-4}
200	2.6875×10^0	1.41	2.50×10^{-2}	4.4×10^{-4}
400	9.1713×10^{-1}	1.55	1.25×10^{-2}	1.5×10^{-3}
800	2.8039×10^{-1}	1.71	6.26×10^{-3}	4.3×10^{-3}
1600	7.3346×10^{-2}	1.93	1.95×10^{-3}	2.2×10^{-2}
3200	1.7622×10^{-2}	2.06	4.88×10^{-4}	9.6×10^{-2}
6400	3.5252×10^{-3}	2.32	1.22×10^{-4}	6.1×10^{-1}

	IMEX			
N	L_1 error	Order	Δt	Time (s)
50	7.8413×10^0	$--$	4.34×10^{-2}	3.8×10^{-4}
100	1.9886×10^0	1.98	2.17×10^{-2}	7.8×10^{-4}
200	5.0056×10^{-1}	1.99	1.09×10^{-2}	2.2×10^{-3}
400	1.2554×10^{-1}	1.99	5.43×10^{-3}	6.9×10^{-3}
800	3.1367×10^{-2}	2.00	2.72×10^{-3}	1.5×10^{-2}
1600	7.7625×10^{-3}	2.02	1.36×10^{-3}	5.0×10^{-2}
3200	1.8499×10^{-3}	2.07	6.80×10^{-4}	1.8×10^{-1}
6400	3.7004×10^{-4}	2.32	3.40×10^{-4}	6.7×10^{-1}
	Explicit			
N	L_1 error	Order	Δt	Time (s)
50	7.4158×10^0	$--$	8.00×10^{-4}	6.7×10^{-3}
100	1.8518×10^0	2.00	2.00×10^{-4}	1.8×10^{-2}
200	4.6253×10^{-1}	2.00	5.00×10^{-5}	8.7×10^{-2}
400	1.1551×10^{-1}	2.00	1.25×10^{-5}	4.8×10^{-1}
800	2.8793×10^{-2}	2.00	3.13×10^{-6}	2.9×10^0
1600	7.1211×10^{-3}	2.02	7.81×10^{-7}	2.0×10^1
3200	1.6999×10^{-3}	2.07	1.95×10^{-7}	1.5×10^2
6400	3.4735×10^{-4}	2.29	4.88×10^{-8}	1.2×10^3

In the left table we show a convection dominated example, while the second table corresponds to a diffusion dominated one. We have checked that the computational times, using IMEX schemes, can be two or three orders of magnitude lower, while retaining the same accuracy and order of convergence.

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