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Second order finite volume IMEX Runge-Kutta schemes for two dimensional parabolic PDEs in finance

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In this work we present a novel and general methodology for building second order finite volume implicit-explicit (IMEX) numerical schemes for solving two dimensional financial parabolic PDEs with mixed derivatives. In particular, applications to basket and Heston models are presented. The obtained numerical schemes have excellent properties and are able to overcome the well-documented difficulties related with numerical approximations in the financial literature. The methods achieve true second order convergence with non-regular initial conditions. Besides, the IMEX time integrator allows to overcome the tiny time-step induced by the diffusive term in the explicit schemes, also providing very accurate and non-oscillatory approximations of the Greeks (the derivatives). Finally, in order to assess all the aforementioned good properties of the developed numerical schemes, we compute extremely accurate semi-analytic solutions using multi-dimensional Fourier cosine expansions. A novel technique to truncate the Fourier series for basket options is presented and it is efficiently implemented using multi-GPUs.

We focus in the solution of the PDE for the price of a basket of two options:

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma_1^2 s_1^2 \frac{\partial^2 u}{\partial s_1^2} - \frac{1}{2}\sigma_2^2 s_2^2 \frac{\partial^2 u}{\partial s_2^2} - \rho\sigma_1\sigma_2 s_1 s_2 \frac{\partial^2 u}{\partial s_1 \partial s_2} - (r-q_1)s_1\frac{\partial u}{\partial s_1} - (r-q_2)s_2\frac{\partial u}{\partial s_2} + ru = 0,\ s_1, s_2 > 0, \tau \in (0,T], \tag{1}$$

with initial condition given by $u(s_1, s_2, 0) = \max(\frac{1}{2}(s_1+s_2) - K, 0)$, where K is the fixed strike price, r is the interest rate, q_i are the dividend yields, σ_i are the volatilities and ρ is the correlation. We also consider the price of a derivative with one underlying option whose price is given by the Heston stochastic volatility model:

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}s^2 v\frac{\partial^2 u}{\partial s^2} - \rho\sigma s v\frac{\partial^2 u}{\partial v \partial s} - \frac{1}{2}\sigma^2 v \frac{\partial^2 u}{\partial v^2} - (r-q)s\frac{\partial u}{\partial s} - \kappa(\theta - v)\frac{\partial u}{\partial v} + ru = 0, \quad s > 0, \tau \in (0,T] \tag{2}$$

with initial condition $u(s, v, 0) = \max(s - K, 0)$, where q is the dividend yield and κ, θ are parameters of the model. In order to build finite volume schemes for Equations (1) and (2), first they must be written in conservative (divergence) form:

$$\frac{\partial}{\partial t}u(x,y,t) + \operatorname{div}\mathbf{F}(u) = \operatorname{div}\mathbf{G}(\nabla u) + h(u), \text{ with } \quad \mathbf{F}(u) = (f_1(u), f_2(u)), \quad \mathbf{G}(\nabla u) = (g_1(\nabla u), g_2(\nabla u)). \tag{3}$$

The numerical solution of equation (3) using a finite volume semi-implicit scheme is difficult because of the presence of the diffusive part. In this work we have considered the IMEX Runge-Kutta time discretization numerical scheme proposed in [1].

Space semi-discretization. Let Δx, Δy be the mesh length in the X, Y directions respectively. We define the grid points $x_i = i\Delta x$, $y_j = j\Delta y$, $i, j \in \mathbb{Z}$. Let $x_{i+1/2} = x_i + \frac{1}{2}\Delta x$ and $y_{j+1/2} = y_j + \frac{1}{2}\Delta y$. We consider rectangular finite volumes $V_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$, where (x_i, y_j) is the center of the finite volume V_{ij}. The area of V_{ij} will be denoted as $|V_{ij}|$, i.e., $|V_{ij}| = \Delta x\Delta y$. Besides, Γ_{ij} represents the boundary of V_{ij}. The unknowns of the problem are the volume averages $\{\bar{u}_{ij}(t)\}$, $\bar{u}_{ij} = |V_{ij}|^{-1}\int_{V_{ij}} u(x,y,t)\mathrm{d}x\mathrm{d}y$. Integrating equation (3) in space on V_{ij} and dividing by $|V_{ij}|$ and applying the divergence theorem, we obtain the semi-discrete equation

$$\frac{\mathrm{d}\bar{u}_{ij}}{\mathrm{d}t} = -\frac{1}{|V_{ij}|}\oint_{\Gamma_{ij}} \mathbf{F}(u)\cdot\mathbf{n}\,d\gamma + \frac{1}{|V_{ij}|}\oint_{\Gamma_{ij}} \mathbf{G}(u_x, u_y)\cdot\mathbf{n}\,d\gamma + \frac{1}{|V_{ij}|}\int_{V_{ij}} h(u)\,\mathrm{d}x\mathrm{d}y. \tag{4}$$

In order to convert (4) into a numerical scheme, we have to approximate on the right hand side of this equation with functions of $\{\bar{u}_{ij}(t)\}$. The source and convective terms will be treated explicitly (see [2, 3]), while the diffusion part will be managed implicitly in time in the IMEX Runge-Kutta scheme (see [1]). For the advection $\int_{\Gamma_{i+1/2,j}} f_1(u)\mathrm{d}\gamma_j \approx \Delta y \mathcal{F}_1(u^-_{i+1/2,j}, u^+_{i+1/2,j})$

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where $\mathcal{F}_1(u^-_{i+1/2,j}, u^+_{i+1/2,j})$ is a suitable numerical flux, evaluated in the reconstructed states at the intercell. At this point the unknown function $u(x,y,t)$ is reconstructed by a piecewise polynomial using the volume averages $\{\bar{u}_{ij}(t)\}$. Second order numerical schemes can be obtained by means of piecewise linear polynomials, although higher order schemes can be obtained by polynomials of higher order. Here we consider the natural extension of MUSCL reconstruction to 2D Cartesian grids with minmod limiters. Finally, the volume integral of the source term can be discretized using the midpoint quadrature rule. For approximating the line integrals related to the diffusion operator, a suitable approximation of the partial derivatives u_x and u_y has to be built for the volume V_{ij}. Because of the presence of mixed derivatives in the second order operator, we build the second order Lagrange interpolating polynomial of u centered in the volume V_{ij}, L^u_{ij}. Using these polynomials, we can compute approximations of the gradients at each intercell, $u_x \approx \partial_x L^{\bar{u}}_{ij}$ and $u_y \approx \partial_y L^{\bar{u}}_{ij}$. The computation of these approximations of the partial derivatives of the solution just involves the computation of derivatives of one dimensional Lagrange polynomial basis. As a summary, the line integrals in equation (4) are approximated as $\int_{\Gamma_{i\pm 1/2,j}} g_1(u_x,u_y)\mathrm{d}\gamma_j \approx \int_{\Gamma_{i\pm 1/2,j}} g_1\Big(\partial_x L^{\bar{u}}_{ij}, \partial_y L^{\bar{u}}_{ij}\Big)\mathrm{d}\gamma_j \approx \pm\Delta y\, g_1(\partial_x L^{\bar{u}}_{ij}(x_{i\pm 1/2}, y_j), \partial_y L^{\bar{u}}_{ij}(x_{i\pm 1/2}, y_j))$,

IMEX Runge-Kutta time discretization. Once we have performed the space discretization, we obtain a stiff ODE system of equations in the form $\frac{\partial U}{\partial t} + E(U) = I(U)$, where $U = U(t) \in \mathbb{R}^N$ and $E, I : \mathbb{R}^N \to \mathbb{R}^N$, being E the non-stiff term and I the stiff part. Both parts will be handled simultaneously with the same IMEX solver. An IMEX Runge-Kutta scheme consists of applying an implicit discretization to the diffusion terms (stiff terms) and an explicit one to the convective and source terms (non stiff terms). A second order IMEX scheme takes the form

$$U^{(k)} = U^n - \Delta t \sum_{l=1}^{k-1} \tilde{a}_{kl} E(U^{(l)}) + \Delta t \sum_{l=1}^{2} a_{kl} I(U^{(l)}), \quad U^{n+1} = U^n - \Delta t \sum_{k=1}^{2} \tilde{\omega}_k E(U^{(k)}) + \Delta t \sum_{k=1}^{2} \omega_k I(U^{(k)}), \tag{5}$$

where $U^n = (\bar{u}^n_{ij})$ is the vectors of unknowns volume averages at times t^n, and $U^{(k)}$ and $U^{(l)}$ are the vector of unknowns at the stages k, l of the IMEX Runge-Kutta scheme. The matrices $\bar{A} = (\tilde{a}_{ij})$, $\tilde{a}_{kl} = 0$ for $l \geq k$ and $A = (a_{kl})$ are 2×2 matrices for the explicit and implicit parts. The coefficient vectors $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)$ and $w = (w_1, w_2)$ complete the IMEX Runge-Kutta tableaus for the scheme, and are given in [1]. The results obtained for a convection dominated test for the Basket model are:

	IMEX			
$N_1 \times N_2$	L_1 error	Order	Δt	Time (s)
25×25	9.9867×10^1	$--$	2.05×10^{-2}	1.1×10^{-2}
50×50	3.3457×10^1	1.57	1.03×10^{-2}	3.9×10^{-2}
100×100	9.1341×10^0	1.87	5.13×10^{-3}	2.3×10^{-1}
200×200	2.3529×10^0	1.95	2.56×10^{-3}	1.8×10^0
400×400	5.6234×10^{-1}	2.06	1.28×10^{-3}	1.6×10^1
800×800	1.1257×10^{-1}	2.32	6.41×10^{-4}	1.5×10^2

	Explicit			
$N_1 \times N_2$	L_1 error	Order	Δt	Time (s)
25×25	9.9752×10^1	$--$	3.56×10^{-2}	9.4×10^{-4}
50×50	3.3453×10^1	1.57	8.88×10^{-3}	1.1×10^{-2}
100×100	9.1503×10^0	1.87	2.22×10^{-3}	8.3×10^{-2}
200×200	2.3614×10^0	1.95	5.56×10^{-4}	1.4×10^0
400×400	5.6466×10^{-1}	2.06	1.39×10^{-4}	2.1×10^1
800×800	1.1304×10^{-1}	2.32	3.47×10^{-5}	3.6×10^2

The results obtained for a diffusion dominated test for the Basket model are:

	IMEX			
$N_1 \times N_2$	L_1 error	Order	Δt	Time (s)
25×25	9.6620×10^1	$--$	4.71×10^{-2}	4.5×10^{-3}
50×50	2.5178×10^1	1.94	2.35×10^{-2}	3.3×10^{-2}
100×100	6.4828×10^0	1.95	1.18×10^{-2}	1.7×10^{-1}
200×200	1.6209×10^0	2.00	5.89×10^{-3}	1.2×10^0
400×400	3.9419×10^{-1}	2.03	2.94×10^{-3}	9.8×10^0
800×800	7.9229×10^{-2}	2.31	1.47×10^{-3}	8.5×10^1

	Explicit			
$N_1 \times N_2$	L_1 error	Order	Δt	Time (s)
25×25	9.0224×10^1	$--$	1.42×10^{-3}	1.7×10^{-2}
50×50	2.3440×10^1	1.94	3.56×10^{-4}	1.2×10^{-1}
100×100	5.9498×10^0	1.97	8.89×10^{-5}	1.8×10^0
200×200	1.4834×10^0	2.00	2.22×10^{-5}	3.0×10^1
400×400	3.5473×10^{-1}	2.06	5.56×10^{-6}	4.9×10^2
800×800	7.1095×10^{-2}	2.31	1.39×10^{-6}	7.9×10^3

Looking at the tables, we have checked that the computational times, using IMEX schemes, can be two or three orders of magnitude lower, while retaining the same accuracy and order of convergence.

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