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Zoom into the shallow: Resolving vertical flow structure with shallow moment equations

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Shallow flow models are used in many geophysical applications like debris flow simulations and weather forecasting, mainly due to their computational efficiency. A classical model are the shallow water equations (SWE), in one space dimension given by

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g_z \frac{h^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ g_x h - g_z h \partial_x h_b \end{pmatrix} \quad , \tag{1}$$

where $h(t,x)$ is the fluid depth, $u_m(x,t)$ the mean flow velocity, $\mathbf{g} = (g_x, g_y, g_z)^T$ the gravitational constant and $h_b(x)$ is the bottom surface elevation. The SWE can be understood as a simplified version of the incompressible Navier-Stokes equations, in case that a free surface flow, kinematic boundary conditions, a hydrostatic pressure closure and the shallowness assumption are considered [1]. The shallowness assumption justifies depth averaging of the vertical velocity field. However, the squared flow velocity $\frac{1}{h}\int_{h_b}^{h_s} u(t,x,z)^2 \mathrm{d}z \approx u_m^2$ is approximated by $u_m(t,x)^2$, which means that the shallow flow equations are exact only when assuming a vertically constant velocity profile, also referred to as plug flow ($h_s(t,x)$ denotes the height of the free surface). A velocity profile that deviates from plug flow, yet is still constant in time, can be accounted for by introducing a shape factor.

Experiments in different geophysical applications have shown that real flow scenarios often exhibit changing velocity profiles during different stages of the flow [2, 3]. Based on such evidence, the community has developed methods to better approximate the vertical velocity fields, see e.g. [4]. In the following a more recent approach, the *Shallow Moment Method* [1], is studied in one space dimension. This method retains information of the vertical velocity profile by splitting the flow velocity

$$u(t,x,z) = u_m(t,x) + u_d(t,x,z) = u_m(t,x) + \sum_{j=1}^{N} \alpha_j(t,x)\,\phi_j(z) \tag{2}$$

into its mean velocity and a deviation $u_d(t,x,z)$, which is approximated by a Legendre expansion with coefficients $\alpha_j(t,x)$ and Legendre polynomials $\phi_j(z)$. The order of expansion N determines the quality of the approximation, e.g. linear velocity profiles for $N = 1$ and quadratic profiles for $N = 2$. By averaging the vertical velocity of the mass and momentum balance of the reference model and subsequent projection of the Legendre polynomials in order to obtain equations for the Legendre coefficients, a system of partial differential equation of order $2 + N$ is constructed. All details regarding the derivation can be found in [1].

For $N = 2$, the system for instance yields

$$\begin{aligned} &\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g_z\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \\ &\qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5u_m - \alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & \frac{7u_m + \alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{R}{\chi}(u_m + \alpha_1 + \alpha_2) + h(g_x - g_z\partial_x h_b) \\ -\frac{R}{\chi}3\left(u_m + \alpha_1 + 4\frac{\chi}{h}\alpha_1 + 3\alpha_2\right) \\ -\frac{R}{\chi}5\left(u_m + \alpha_1 + \alpha_2 + 12\frac{\chi}{h}\alpha_2\right) \end{pmatrix} \end{aligned} \quad , \tag{3}$$

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where R is the friction coefficient and χ the slip length. The system's first and second equation correspond to mass and momentum balance, which are affected by higher order moments. All additional rows of the system provide evolution equations for these moments. In (3), different colors refer to different levels of the moment hierarchy, namely black: zeroth order, black+red: first order and black+red+blue: second order. Remark that the classical SWE (1) are recovered by choosing $N = 0$ and neglecting friction ($R = 0$). A close look at the first and second order model reveals that the parameters R and χ are the driving forces for non-constant velocity profiles, since line three and four in (1) remain zero if the initial velocity profile is constant and $R = 0$.

At last, a test case studying the interaction of vertical velocity profiles between colliding waves is presented. The flow of a release mass (box function in figure (1)) along a plane rotated at an angle of 15 degree downwards in positive x-direction is simulated. The initial velocity is zero everywhere with parameters $(R, \chi) = (0.075, 0.1)$, periodic boundary conditions and no bottom topology $h_b(x) = 0$. The velocity profile of the first plot figure (2) corresponds to the tracers of wave propagating to the

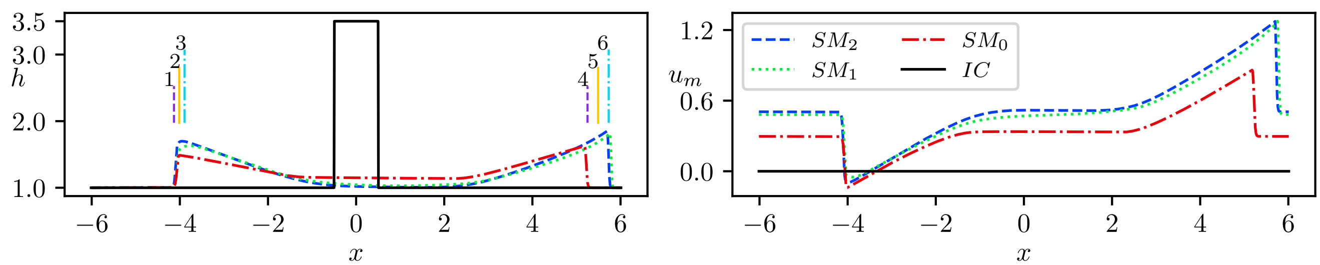

Figure 1: Fluid depth and mean velocity at $T = 2.91$ for a zeroth, first and second order model. The vertical lines indicate the positions of the vertical velocity profiles shown in figure (2) for the second order model.

left in figure (1). The profiles show that there are rapid changes over small distances due to the interaction of waves propagating to the left and right. Note that in a frictionless scenario, all velocity profiles would remain constant. In the second plot, where the wave front is propagating to the right, all shapes are similar and differ only in magnitude due to the position of the wave front.

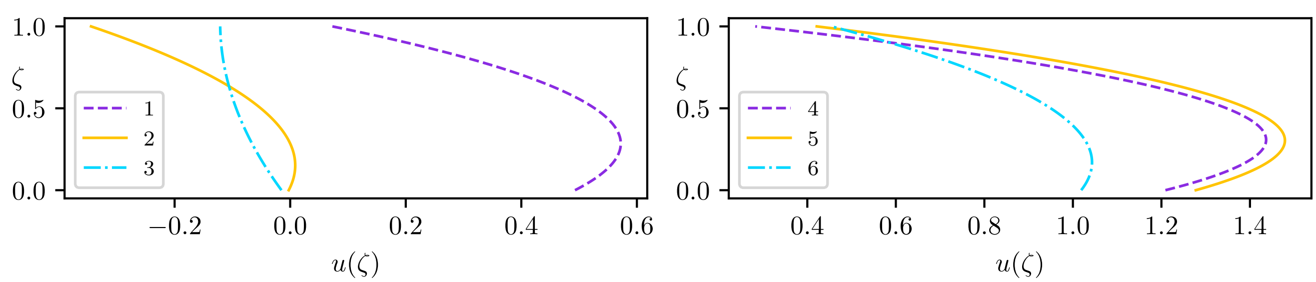

Figure 2: Velocity profiles corresponding to the indicated positions in figure (1).

As a next step, we will investigate the effect of complex bottom topography incorporating different friction mechanisms and extend our results to two space dimensions.

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