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Dispersive Moment Equations for Shallow Flow

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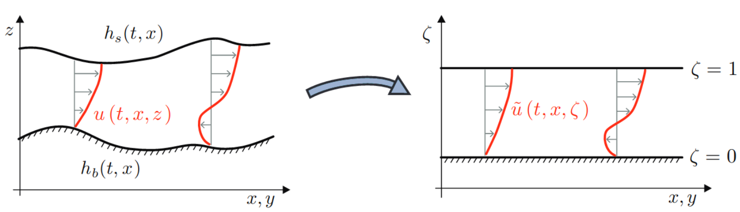
Shallow flow models are used in situations where a flow's vertical extent is much smaller than its horizontal extent. These models exploit the shallowness by depth-averaging, which leads to the fact that they have lower computational costs than a corresponding vertically resolved free-surface flow. Quite naturally, the simpler formulation comes at the price of loosing vertical information, such as information on the velocity profile. However, in many realistic situations, an approximation of the velocity profile is needed. One approach to solve this problem is the usage of vertical moments as suggested by J. Kowalski and M. Torrilhon in [1]. On the other hand, dispersive models have gained popularity in the recent years. They are capable of capturing phenomena like solitons and cnoidal waves and yield overall better descriptions of wave behaviour in coastal areas. In our work, we combine both approaches and suggest a framework for the systematic derivation of equation systems that model nontrivial vertical profiles of the flow variables and simultaneously recreate the dispersive behaviour of the fully resolved equations.

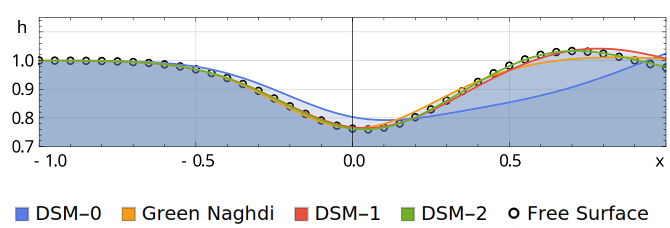


Ansatz

$$\begin{bmatrix} u(x,\zeta) \\ w(x,\zeta) \\ q(x,\zeta) \end{bmatrix} = \begin{bmatrix} u_m(x) \\ w_m(x) \\ q_m(x) \end{bmatrix} + \sum_{j=1}^{N} \begin{bmatrix} \alpha_j(x)\phi_j(\zeta) \\ \gamma_j(x)\phi_j(\zeta) \\ \kappa_j(x)\phi_j(\zeta) \end{bmatrix}$$

We call our equations the Dispersive Shallow Moment Equations (DSM). They rely on a polynomial ansatz and subsequent Galerkin projection in the incompressible Euler equations for free surface flow in a transformed coordinate system where a new ζ-variable replaces the old vertical z-variable (see picture). In contrast to [1] the equations include a non-hydrostatic pressure component q. Therefore, the systems are dispersive, similar to the well-known Green-Naghdi-Equations [2].

For a fixed height-to-depth ratio S the dispersion relation of the vertically resolved free surface flow is approximated up to arbitrary precision by the DSM models, depending on the degree of the polynomial ansatz N. In order to show pracitical relevance we present results from a two-dimensional numerical test case. We see that the DSM models can compete with the Green-Naghdi system in terms of cost-effectiveness and are more accurate for higher degrees of the polynomial ansatz N.



Finally, we show that for a fixed polynomial degree N our equation systems converge to the classical shallow flow equations as we let the height-to-depth ratio S go to zero.

References

[1] Julia Kowalski and Manuel Torrilhon. "Moment Approximations and Model Cascades for Shallow Flow", *Commun. Comp. Phys.*, 25(3), pp. 669–702, 2019.

[2] Green, A. E. and Naghdi, P. M.. "A derivation of equations for wave propagation in water of variable depth", *Journal of Fluid Mechanics*, 78(2), pp. 237–246, 1976.

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