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Weak-strong stability for wave-maps

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This presentation is concerned with the stability analysis of wave maps and its use for analyzing numerical approximations. Wave maps are semi-linear wave equations with the point-wise constraint that the solution takes values in some given target manifold. They arise as critical points of a Lagrange functional for manifold valued functions and serve as model problems in general relativity and in particle physics. In the case that the target manifold is a 2-sphere they can be written as: Find $u : \Omega \times (0, T) \to S^2$ such that

$$\partial_{tt} u - \Delta u = (|\nabla u|^2 - |\partial_t u|^2)u$$
$$\text{with point-wise constraint } |u(t, x)| = 1.$$

and need to be complemented by initial data for u and $\partial_t u$ and suitable boundary conditions.

Strong solutions of wave-maps satisfy an energy conservation principle. Their long time behavior has received a lot of attention and depending on the dimension, the target manifold and the size of the initial data strong solutions may exist for all times or there might be gradient blow-up at some finite time [2]. After gradient blow-up, only weak solutions exist and it is known that they are not unique if no energy dissipation criterion is enforced [4]. If attention is restricted to (energy dissipative) finite energy weak solutions it is unknown whether (or under which circumstances) these solutions are unique.

What is known regarding uniqueness (for arbitrary dimensions and target manifolds) is weak-strong uniqueness, i.e. sufficiently strong solutions are (as long as they exist) unique in the class of finite energy weak solutions [5].

We have modified the argument leading to weak strong-uniqueness such that we obtain weak-strong stability. This new result is very similar to relative entropy estimates in systems of hyperbolic conservation laws and it allows us to quantify the difference between solutions in terms of differences in initial data and we can also include non-homogeneous right hand sides in the equation satisfied by one of the solutions.

We use this stability framework to derive a posteriori error estimates for the numerical scheme proposed in [6]. The error estimators, i.e. upper bounds for the error in the energy norm that can be computed from the numerical solution, provide useful bounds until there is gradient-blow up in the numerical solution [3]. This is reminiscent to results that we have obtained for numerical approximations of systems of hyperbolic conservation laws [1].

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