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Existence of solutions of the initial boundary value problem for a non-strictly hyperbolic system of conservation laws

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We consider Initial Boundary Value Problems (IBVP) consisting in 2×2 hyperbolic systems of conservation laws, namely

$$
\begin{aligned}
&\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v) = 0, \end{cases} && x \in]x_{in}, x_{out}[\subset \mathbb{R},\ t > 0, \\
&(\rho, w)(0, x) = (\rho_0, w_0)(x), && x \in]x_{in}, x_{out}[, \\
&(\rho, w)(t, x_{in}) = (\rho_{in}, w_{in})(t), && t \in]0, T[, \\
&(\rho, w)(t, x_{out}) = (\rho_{out}, w_{out})(t), && t \in]0, T[,
\end{aligned}
\tag{1}
$$

with values in an invariant domain of the form

$$\Omega := \left\{(\rho, w) \in \mathbb{R}^2 \colon \rho \in [0, R(w_{max})], w \in [w_{min}, w_{max}], 0 < w_{min} \leq w_{max} < +\infty\right\}$$

and closed by a given relation $v = \mathcal{V}(\rho, w)$.
General well-posedness results for Initial Boundary Value Problems (IBVP) provided in the literature hold under the hypothesis of strict hyperbolicity and eigenvalues with constant signs (see e.g. [3, Theorem 2.3]). However, since the system in (1) equipped with eigenvalues

$$\lambda_1(\rho, w) = \mathcal{V}(\rho, w) + \rho \mathcal{V}_\rho(\rho, w), \qquad \lambda_2(\rho, w) = \mathcal{V}(\rho, w),$$

is only strictly hyperbolic for $\rho > 0$ and has an eigenvalue changing sign, we propose an existence proof for this specific problem.

In the literature, two definition of boundary conditions for systems of conservation laws are considered [4]: an *entropy boundary inequality* derived by viscosity approximation and a *Riemann boundary condition* based on the Riemann solver. For problem (1), the two definitions do not coincide. Thus, we study the admissibility of the states on the boundary, proving that the admissible states defined by the *Riemann boundary set* satisfy the boundary entropy condition. Following [1, 2], we can then prove that approximate solutions, constructed by wave-front tracking, converge to a weak entropy solution of the IBVP (1) with initial and boundary data of bounded variation.

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