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Global in time existence of solutions with L^1-initial data for the revised Enskog equation

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The revised Enskog equation generalizes the Boltzmann equation to moderately dense gases. Originally proposed by Enskog [1], and in 1973 modified by H. van Beijeren and M. H. Ernst ([2], it generalizes Boltzmann's original *stosszahlansatz* in two ways:

1. by taking into account the fact that the centers of two colliding spheres are at a distance a, equal to the diameter of hard spheres.
2. by increasing the collision frequency by a factor Y_0 which nowadays is identified with the radial pair correlation function $g_2(r)$ for the system of hard spheres at a **uniform** equilibrium.

The advantage of considering hard sphere systems resides in two facts: the collisions are instantaneous and influence of multiple collisions (i.e. simultaneous encounters of more than two spheres) is negligible. In fact, the volume of the phase space corresponding to triple, quadruple, and n-tuple collisions is zero; at least for integrable functions.

Furthermore, in moderately dense gases the molecular diameter is no longer small compared with the mean free path between collisions. An important consequence of this is that the transport of momentum and energy during collisions (negligible in the dilute-gas limit, and consequently in the Boltzmann equation) takes place over distances comparable to the separation of the molecules.

H. van Beijeren and M. H. Ernst ([2]) modified the original Enskog theory by replacing the radial pair correlation function $g_2(r)$ at uniform equilibrium by the exact pair correlation function $g_2(r_1, r_2 \,|\, n)$ that takes full account of spatial non-uniformity in the local equilibrium. In contrast to the Boltzmann equation with hard-sphere potential, in equilibrium, the solutions of this revised Enskog equation reduce to an exact density functional description of the state of hard spheres in a local equilibrium.

The Mayer cluster expansion of $g_2(r_1, r_2 \,|\, n)$ has the form

(1)
$$\begin{aligned} g_2(r_1, r_2 \,|\, n) = \exp(-\beta\phi^{HS}(|r_1 - r_2|))\Big\{1 + \int V(12\,|\,3) n(t, r_3)\, dr_3 + \frac{1}{2}\int\int V(12\,|\,34) n(t, r_3)\, n(t, r_4)\, dr_3 dr_4 \\ + \cdots + \frac{1}{(k-2)!}\int dr_3 \cdots \int dr_k\, n(3) \cdots n(k)\, V(12\,|\,3 \dots k) + \cdots \Big\} \end{aligned}$$

where $n(k) = n(t, r_k),\ \beta = 1/k_B T,\ V(12\,|\,3 \dots k)$ is the sum of all graphs of k labeled points which are biconnected when the Mayer factor $f_{ij} = \exp(-\beta\phi^{HS}(|r_i - r_j|)) - 1 = \Theta(|r_i - r_j| - a) - 1$ is added. Here, Θ is the step function and ϕ^{HS} is the hard-sphere potential:

$$\phi^{HS}(r) = \begin{cases} +\infty, & \text{if } r \le a; \\ 0, & \text{if } r > a, \end{cases} \qquad \text{and } a \text{ equal to the diameter of hard spheres.}$$

For example, if $\Omega \subseteq \mathbb{R}^3$ denotes the spatial domain, the second term in (1) has the form

(2)
$$\int n(t, r_3)\, V(12\,|\,3)\, dr_3 = \int n(t, r_3)\, f_{13}\, f_{23}\, dr_3 = \int\limits_{\Omega \cap \left\{\substack{|r_1 - r_3| \le a \\ |r_2 - r_3| \le a}\right\}} n(t, r_3)\, dr_3,$$

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The revised Enskog equation for the one–particle distribution function $f(t,r,v)$ is a nonlinear integro-partial differential equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial r} = E^+(f) - E^-(f), \quad r\in\Omega\subseteq\mathbb{R}^3,\, v\in\mathbb{R}^3, \tag{3}$$

with

$$E^+(f) = a^2 \iint_{\mathbb{R}^3\times\mathbb{S}^2_+} g_2(r, r-a\epsilon\,|\,n(t)) f(t,r,v')\, f(t,r-a\epsilon,w')\,\langle\epsilon\cdot, v-w\rangle\, d\epsilon dw, \tag{4}$$

$$E^-(f) = a^2 \iint_{\mathbb{R}^3\times\mathbb{S}^2_+} g_2(r, r+a\epsilon\,|\,n(t)) f(t,r,v)\, f(t,r+a\epsilon,w)\,\langle\epsilon\cdot, v-w\rangle\, d\epsilon dw, \tag{5}$$

where $\langle\cdot,\rangle$ is the inner product in $\mathbb{R}^3$, $\mathbb{S}^2_+ = \{\epsilon\in\mathbb{R}^3 : |\epsilon| = 1, \langle\epsilon, v-w\rangle \geq 0\}$, and $v' = v-\epsilon\langle\epsilon, v-w\rangle$, $w' == w+\epsilon\langle\epsilon, v-w\rangle$ are post-collisional velocities.

The only global existence result for (3) with L^1-initial data was obtained by Arkeryd and Cercignani [3] in the case of the truncated $g_2(r_1, r_2\,|\,n) = \exp(-\beta\phi^{HS}(|r_1-r_2|))$ (see (1)) corresponding to the contact value of $g_2 = 1$ and with **NO** local density dependence. Such a truncated kinetic equation is often called the Boltzmann-Enskog equation. It differs from the Boltzmann equation only by existence of the shifts in the spatial variable in the collisional integral.

The convergence of the infinite Mayer cluster expansion of $g_2(r_1, r_2\,|\,n)$ in (1) is not known at this moment, except in the homogeneous case (n **not** depending on the spatial variable and thus not applicable here), when the series converges for small n. Instead, for any $N\geq 1$, I consider equation (3) with a finite, but arbitrary, Mayer cluster expansion of $g_2(r_1, r_2\,|\,n)$:

$$\begin{aligned} g_2(r_1, r_2\,|\,n) = \exp(-\beta\phi^{HS}(|r_1-r_2|))\Big\{ 1 + \int V(12\,|\,3) n(t,r_3)\, dr_3 + \frac{1}{2}\int\int V(12\,|\,34) n(t,r_3)\, n(t,r_4)\, dr_3 dr_4 \\ + \cdots + \frac{1}{(N-2)!}\int dr_3 \cdots \int dr_N\, n(3)\cdots n(N)\, V(12\,|\,3\ldots N)\Big\}, \end{aligned} \tag{6}$$

where $n(k) = n(t, r_k)$ and $1\leq k\leq N$ and prove global in time existence of weak solutions with L^1-initial data in the spatial domain $\Omega = \mathbb{R}^3$ or $\Omega = [0,L]^3$ with periodic boundary conditions. Dependence of g_2 in (6) on n requires a different approach and new tools as compared to Arkeryd-Cercignani's proof in [3].

The proof of existence of weak solutions to (3) is based on two constructions.

1. Construction of an H-functional in the form (see also [4], where the full expansion (1) of $g_2(r_1, r_2\,|\,n)$ was used, but convergence of the series was not addressed):

$$H_E = \iint f(t,r,v)\log f(t,r,v)\, dv dr - \sum_{k=2}^{N}\frac{1}{k!}\int dr_1 \cdots \int dr_k\, n(1)\ldots n(k)\, V(1\ldots k),$$

where $V(1\ldots k)$ is the sum of all irreducible Mayer graphs which doubly connect k particles. For example, with the Mayer factor $f_{ij} = \Theta(|r_i - r_j| - a) - 1$ and Θ being the step function,

$$\int n(t,r_1)\, n(t,r_2)\, V(12)\, dr_1 dr_2 = \int n(t,r_1)\, n(t,r_2)\, f_{12}\, dr_1 dr_2 \quad \text{and}$$

$$\int n(t,r_1)\, n(t,r_2)\, n(t,r_3)\, V(123)\, dr_1 dr_2 dr_3 = \int n(,tr_1)\, n(t,r_2)\, n(t,r_3)\, f_{12}f_{23}f_{13}\, dr_1 dr_2 dr_3\,.$$

2. Construction of a special sequence of stochastic kinetic equations (studied in [5]) and a proof that their solutions converge to weak solutions of the revised Enskog equation (3).

References

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