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A well-balanced relaxation method for low Mach number flows with gravity.

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In physical applications, problems with large scale differences can occur, so that the Mach number becomes very small. It is commonly known that for such problems standard finite volume schemes suffer from excessive dissipation. This can be explained by analysing the structure of numerical fluxes in finite volume methods. In general, finite volume fluxes consist of a central flux combined with an artificial dissipation term, i.e.

$$\boldsymbol{F}(\boldsymbol{U}^L, \boldsymbol{U}^R) = \frac{1}{2}\left(\boldsymbol{\mathcal{F}}(\boldsymbol{U}^L) + \boldsymbol{\mathcal{F}}(\boldsymbol{U}^R)\right) - \frac{1}{2}\boldsymbol{D}(\boldsymbol{U}^R - \boldsymbol{U}^L). \tag{1}$$

This dissipation term, introduced for stability, scales with the largest wave speed of the underlying system and therefore with the inverse of the Mach number, i.e. $\boldsymbol{D} \sim \mathcal{O}(1/M)$. In combination with the difference between left and right state of the velocity at the Riemann problem, this leads to a very large dissipation which prevents an accurate resolution of the fluid flow. Based on this problem analysis, low Mach fixes were introduced for various approximate Riemann solvers to reduce dissipation. In low Mach versions of Roe's solver, the dissipation matrix is rescaled by multiplying by carefully chosen preconditioning matrices [1, 2], while in HLL-type solvers it is sufficient to redefine the intermediate state of the pressure [3]. When incorporating this low dissipation strategy into the theory of Suliciu type relaxation solvers as well, the subcharacteristic condition for stability has to be taken into account. Unfortunately, it is not possible to simply rescale the already existing relaxation speed to reduce dissipation, as this would violate the subcharacteristic condition. To circumvent this conflict, Bouchut *et al.* add a second relaxation speed to their relaxation system for solving the homogeneous barotropic Euler equations [4]. The key idea here is to not only reduce dissipation on the velocity, but to simultaneously increase dissipation on the density. So, in a way, a transfer of dissipation takes place. As a result, the dissipation remains bounded for low Mach numbers, while the subcharacteristic condition remains fulfilled. The additional dissipation on the density does not result in a reduced accuracy since the density difference scales with M^2 and thus the dissipation remains bounded.

We extend this low Mach approach to the full Euler equations and consider additionally gravitational source terms [5]. In this context one must also consider the influence of the source term on steady solutions. For problems close to hydrostatic equilibrium

$$\begin{cases} \boldsymbol{u} = 0, \\ \nabla p = -\rho \nabla \Phi, \end{cases} \tag{2}$$

standard finite volume methods do not automatically satisfy a discrete equivalent of (2) and therefore are not capable of resolving small perturbations on the equilibrium accurately on coarse grids. In order to adress this we combine the two-speed system with a well-balancing mechanism that was introduced in [6]. The key idea of this approach is to add a transport relaxation equation for the gravitational potential to the relaxation system, which leads to a Riemann problem that is under-determined. This gives an additional degree of freedom and allows to introduce a closure equation that is a discrete equivalent of (2) and ensures the well-balanced property. The resulting approximate Riemann solver is proven to be well-balanced and asymptotic preserving. In the asymptotic-preserving proof, it becomes very clear that the well-balanced property is closely related to the asymptotic preserving property and, in a sense, is even a prerequisite for it. By respecting the subcharacteristic condition, the solver satisfies a discrete version of the entropy inequality, with the help of which it can be proven that no unphysical checkerboard modes can occur in the velocity and the pressure. Additionally, the approximate Riemann solver preserves the positivity of density and internal energy. These properties can also be substantiated in numerical tests.

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