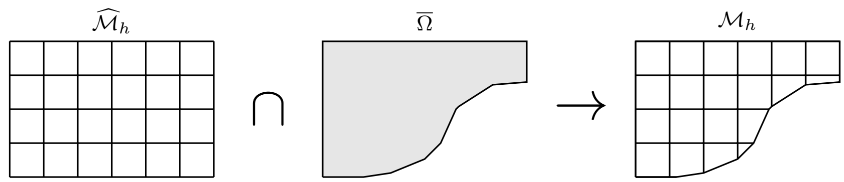
XVIII International Conference on Hyperbolic Problems
Theory, Numerics, Applications
Málaga, June 20-24, 2022

DoD-stabilized DG schemes for solving conservation laws on cut cell meshes

F. Streitbürger*, G. Birke †, C. Engwer ‡, S. May §

Cut cell meshes are an excellent way to drastically speed up the mesh generation process for complex geometries. The concept is quite simple: One constructs the cut cell mesh $\mathcal{M}_h$ by *cutting* a given geometry $\overline{\Omega}$ out of an easily generated, Cartesian background grid $\widehat{\mathcal{M}}_h$ as illustrated in the following, compare [1]:



The downside is that the resulting cut cells, which can become very small, cause standard schemes to become unstable. As a result one has to modify these methods appropriately to guarantee stability on small cut cells. When solving hyperbolic conservation laws on cut cell meshes the main difficulty is the so-called small cell problem: For explicit time stepping, if one chooses the step length based on the size of the larger background cells, one typically encounters stability problems on the small cut cells.

In this talk, we present a penalty stabilization, called Domain of Dependence (DoD) stabilization, for stabilizing discontinuous Galerkin (DG) schemes in space. The DoD stabilization solves the small cell problem by adding suitable penalty terms to the standard discretization. The resulting DoD stabilized semi-discrete scheme is then given by: Find $u^h(t) \in V_h^p$ such that

$$(1) \qquad \left(d_t u^h(t), w^h\right)_{L^2} + a_h(u^h(t), w^h) + J_h(u^h(t), w^h) = 0 \qquad \forall w^h \in V_h^p,$$

with V_h^p being the discrete function space of piecewise polynomials of degree p. Further, $a_h(\cdot,\cdot)$ is the standard DG discretization of the convection term and $J_h(\cdot,\cdot)$ is the collection of DoD stabilization terms.

The DoD stabilization consists of two components $J_h(\cdot,\cdot) = J_h^0(\cdot,\cdot) + J_h^1(\cdot,\cdot)$, which aim for different goals. The first term $J_h^0(\cdot,\cdot)$ redistributes the mass in the neighborhood of the small cells in a more physical way. This happens through additional fluxes between the inflow and outflow neighbors of the small cut cells. This stabilization term is given as an interface term. The redistribution of mass restores the proper domains of dependencies, not only for the small cut cells but also for their neighbors. The second term $J_h^1(\cdot,\cdot)$ corrects the mass distribution within the cells in the neighborhood of the small cut cells. It is given as a volume term.

The DoD stabilized scheme (1) can be combined with a standard explicit time stepping and is stable for a time step length that is independent of the size of the small cut cells. Additionally, one can prove that the given scheme is monotone for solving scalar equations using piecewise constant polynomials V_h^0 and explicit Euler in time. In the semi-discrete setting it is possible to show an L^2 stability result for scalar conservation laws in one dimension for general polynomial degrees p [2].

We will give an overview of the DoD stabilization in one and two dimensions. In the one-dimensional case, we will explain the idea of the DoD stabilization for the simplified model problem of linear advection. We will briefly sketch how to extend the

*Department of Mathematics, TU Dortmund University, Vogelpothsweg 87, 44227 Dortmund, Germany, florian.streitbuerger@math.tu-dortmund.de
†Department of Mathematics, WWU Münster, Einsteinstrasse 62, 48149 Münster, Germany, g_birk01@uni-muenster.de
‡Department of Mathematics, WWU Münster, Einsteinstrasse 62, 48149 Münster, Germany, christian.engwer@uni-muenster.de
§Department of Mathematics, TU Dortmund University, Vogelpothsweg 87, 44227 Dortmund, Germany, sandra.may@math.tu-dortmund.de

scheme to non-linear systems. We will present numerical results for non-linear problems and higher-order polynomials. In the second part, we will show how to extend the DoD stabilization for solving the advection equation in two dimensions. We will conclude the talk with numerical results in two dimensions and an outlook to the next steps.

Acknowledgements

This work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through grant 439956613 (Hypercut).

References

[1] C. Engwer, S. May, C. Nüßing, F. Streitbürger. A stabilized discontinuous Galerkin cut cell method for discretizing the linear transport equation. *SIAM J. Sci. Comput.*, 42(6):A3677–A3703, 2020.

[2] S. May, F. Streitbürger. DoD stabilization for non-linear hyperbolic conservation laws on cutcell meshes in one dimension. *Appl. Math. Comput.*, 419, 2021.