XVIII INTERNATIONAL CONFERENCE ON HYPERBOLIC PROBLEMS
THEORY, NUMERICS, APPLICATIONS
Málaga, June 20-24, 2022

High order commutator estimates for SPDEs with transport-type noise

Helge HOLDEN [*], Kenneth H. KARLSEN [†] and Peter H.C. PANG[‡]

In their celebrated paper of 1989, DiPerna and Lions [2] proved a "folklore lemma" on the commutator of multiplication by a $(W^{1,\beta}(\mathbb{R}^d))^d$ function and mollification by a standard Friedrichs mollifier, acting on a function $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$. This result has fundamental in establishing many results for the existence and uniqueness of renormalised solutions in transport type equations for rough coefficients since, and remains important today (see, e.g. [1] and the works cited in and then inspired by it).

Convolution of a transport-type equation understood in a weak sense in the spatial variables against a standard Friedrichs mollifier allows us to understand that equation pointwise in space, for functions that, having been mollified, are smooth. This procedure makes permissible many back-of-the-envelope calculations that would otherwise be unjustified. Commutators are introduced as error terms in the process of mollification to preserve the transport structure of the equation, which must then be controlled.

Consider now a stochastic PDE with transport type noise, i.e., noise of the form $\nabla \cdot (\sigma(x)u) \circ \mathrm{d}W$, where u is the unknown and σ is a given, sufficiently smooth function, and W is a standard Brownian motion. Just as a transport structure in a PDE necessitates the commutator estimate of DiPerna–Lions, controlling a transport-type Stratonovich noise into an Itô type equation requires *double* commutator estimates. This was encountered in Punshon-Smith–Smith's work on a stochastic Boltzmann equation under a divergence-free condition on σ [5]. In a recent joint work with H. Holden and K.H. Karlsen on stochastic Hunter–Saxton and Camassa–Holm equations, we had again to control this double-commutator without a divergence-free condition, but in one spatial dimension [3, 4].

In this talk, I shall report on some results concerning higher order commutator estimates without the divergence-free condition, in the context of a stochastic nonlocal water wave equation.

Acknowledgements

This research was jointly and partially supported by the Research Council of Norway Toppforsk project *Waves and Nonlinear Phenomena* (250070) and the Research Council of Norway project *Stochastic Conservation Laws* (250674/F20).

References

[1] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. math.,* **158** (2004) 227 – 260.

[2] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98(3)** (1989), 511–547.

[3] H. Holden, K. H. Karlsen, and P. H.C. Pang. The Hunter–Saxton equation with noise. *J. Differential Equations*, **270** (2021), 725–786.

[4] H. Holden, K. H. Karlsen, and P. H.C. Pang. Global well-posedness of the viscous Camassa–Holm equation with gradient noise. *(in preparation)*

[5] S. Punshon-Smith and S. Smith. On the Boltzmann equation with stochastic kinetic transport: global existence of renormalized martingale solutions. *Arch. Ration. Mech. Anal.*, **229(2)** (2018), 627–708.

[*]Department of Mathematical Sciences, NTNU, NO-7491 Trondheim, Norway. Email: helge.holden@ntnu.no
[†]Department of Mathematics, University of Oslo, NO-0316 Oslo, Norway. Email: kennethk@math.uio.no
[‡]Department of Mathematical Sciences, NTNU, NO-7491 Trondheim, Norway. Email: peter.pang@ntnu.no