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A moving mesh Lagrangian Finite Volume method for convection-diffusion equations

Niklas KOLBE [*], Hirofumi NOTSU [†]

Partial differential equations have become an important tool in the study of bio-medical problems. Convection-diffusion equations in particular have been extensively used in the modeling of cell migration processes which play crucial roles in applications such as wound healing and cancer progression [1, 2, 3]. Many of these models have in common that regions of highly concentrated densities occur simultaneously with diffuse profiles. In some cases, the dynamics include occurring peaks that emerge in a smooth solution, split, and merge with each other [4]. These phenomena while being important to study in detail are particularly challenging to resolve numerically in an efficient way. In the previous work [5] a scheme was constructed that addresses this problem in 1D. By considering the problem in transformed variables the method can resolve areas of high concentrations accurately by following the mass-transport approach. In this talk we introduce a new scheme based on the Lagrange-Galerkin method (see, e.g., [6]) building on the previous approach. The scheme allows for applications on domains of higher dimensions while resolving highly concentrated regions accurately.

The scheme is applied in case of convection dominated convection-diffusion problems and introduced for the generic equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u}\,\rho) = f(\rho) + \nu \Delta \rho \quad \text{in } \Omega \times (0, T)$$

that includes linear diffusion and reaction. We employ an adaptive mesh consisting of either interval mesh cells in 1D or triangle cells in 2D. Such a mesh can be represented by a set of vertices that includes the cell interfaces in 1D or the triangle vertices in 2D. In our approach we move the mesh by operating on these vertices. The full scheme employs a piece-wise constant finite volume discretization and consists of three stages that are subsequently performed after initialization until the final time of iteration is reached.

In the *mesh update* the vertices of the mesh are moved along the characteristic. In more details the term $\Delta t \mathbf{w}$ is added to each vertex coordinate, where $\mathbf{w}$ considers both convection according to the velocity $\mathbf{u}$ and diffusion. The latter is taken into account in a new way by using a description in linear basis functions. We allow for h-refinement after the movement, where mesh cells are bisected if a given maximum cell size (in 1D) or inner angle (in 2D) is exceeded. In the following *convection update* the isolated convection equation is solved on the updated mesh following the Lagrange-Galerkin approach. To this end the numerical solution of the previous time step is integrated over shifted updated mesh cells. The shifting is done by following the characteristics back in time. Combining the mesh update and the shifting in this way leads to a reduced search area employed in the Lagrangian integration. Eventually we employ the G-method [7] in the *diffusion update*, which accurately resolves the diffusion equation on the modified mesh.

We show various theoretical properties of the scheme. In particular we first show that our mesh updates never lead to self-intersecting and hence invalid meshes if suitable time increments are chosen. In the 1D-case a purely discrete condition on the time increment is provided. Moreover for the diffusion free case we prove existence of an approximate solution, its stability and conservation of mass as well as error estimates. Key in the proofs is a new estimate on the time dependent interpolation operator.

We applied the scheme in various numerical experiments. Experimentally we observed the expected first order of convergence in space, even increasing to second order when small time steps were used. We obtained a significant improvement over the uniform scheme without mesh movement in the L^1, the L^2 and the L^∞ error. Further minor error reductions were obtained when

[*]Institute of Geometry and Practical Mathematics, RWTH Aachen University, Germany. Email: kolbe@igpm.rwth-aachen.de
[†]Faculty of Mathematics and Physics, Kanazawa University, Japan. Email: notsu@se.kanazawa-u.ac.jp

h-refinement was used in addition. We could moreover see a benefit over uniform schemes, when we applied the new scheme to a model of cancer invasion of tissue that features dynamic interactions of peaks.

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