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An efficient shallow model for granular avalanches with a weakly non-hydrostatic pressure

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Granular flows have been widely studied in recent years, since they play an important role in the study of natural hazards (avalanches, submarine landslides,...) and industrial processes. The understanding of these flows is still a challenge from several aspects: the definition of rheological laws describing its complex dynamics (e.g. solid-fluid transitions); the mathematical modelling of these flows is still a difficult task, in particular due to the complex definition of the stress tensor contributions.

Concerning the physical behaviour, the $\mu(I)$-rheology [4] is the most accepted constitutive law describing these flows nowadays. This rheology considers a pressure and strain rate dependent viscosity by means of a variable friction coefficient

$$\mu = \mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0 + I}\, I,$$

where μ_s, μ_2, I_0 are constant values, and I is the inertial number defined as

$$I = \frac{2d_s\|D\|}{\sqrt{p_T/\rho_s}}.$$

In previous equation, p_T is the pressure, d_s is the particle diameter and ρ_s, φ_s are the particle density and solid volume fraction, respectively. D is the strain-rate tensor and $\|D\| = \sqrt{0.5D : D}$.

Shallow models including the $\mu(I)$-rheology have been recently proposed, both single layer [3] and multilayer shallow models [1], in the hydrostatic framework. We derive here a shallow model, which is an extension of these models, by considering a weakly non-hydrostatic pressure in the sense that we do not take into account all the non-hydrostatic contributions. Thus, considering non-dimensional variables, the following expansion of the total pressure (p_T) is assumed:

$$p_T = \frac{\rho}{Fr^2}\,(b + h - z) + \varepsilon q_1 + \varepsilon^2 q,$$

where $\rho = \rho_s\varphi_s$ is the apparent flow density, $\varepsilon = H/L$ is the usual shallowness parameter, Fr the Froude number and b the bottom topography in tilted coordinates. Finally, q_1 and q are the first and second order terms of the non-hydrostatic counterpart, respectively. Then, the non-dimensional vertical momentum conservation equation reads

$$\rho\varepsilon^2\big(\partial_t w + u\,\partial_X w + w\,\partial_Z w\big) + \varepsilon\partial_Z q_1 + \varepsilon^2\partial_Z q = \varepsilon\partial_X(\tau_{ZX}) + \varepsilon\partial_Z(\tau_{ZZ}).$$

Now, by comparing the terms with same order of magnitude in previous equation, we obtain the terms related to the stress tensor

$$\partial_Z q_1 = \partial_X(\tau_{ZX}) + \partial_Z(\tau_{ZZ}), \tag{1}$$

and the ones representing the vertical acceleration of the flow

$$-\,\partial_Z q = \rho\big(\partial_t w + u\,\partial_X w + w\,\partial_Z w\big). \tag{2}$$

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Our goal here is to derive the simplest depth-averaged non-hydrostatic model, improving the results of hydrostatic models. Therefore we consider a model that neglects first order terms (1) and keeps the second order contributions (2) of the non-hydrostatic pressure. The main reason is that it is a widely studied problem for shallow water flows, namely from the numerical point of view, and the numerical results show that this choice significantly improves the results compared to the hydrostatic model.

After the usual integration procedure to obtain depth-integrated models in tilted coordinates over a reference plane with constant slope θ, the final system reads

$$\begin{cases} \partial_t h + \partial_X (hu) = 0, \\ \rho\Big(\partial_t (hu) + \partial_X \left(hu^2\right) + g\cos\theta h \partial_X \left(h + b + \widetilde{b}\right)\Big) = -\big(\partial_X (hq) + 2q\partial_X b\big) - \tau_{XZ|_b}, \\ \rho\big(\partial_t (hw) + \partial_X (huw)\big) = 2q, \\ hw = hu\partial_X b - \dfrac{h}{2}\partial_X hu + \dfrac{hu}{2}\partial_X h, \end{cases}$$

where $\widetilde{b}(X) = -X\tan\theta$ is the reference plane and

$$\tau_{XZ|_b} = \begin{cases} \mu(I_{|_b}) p_{T|_b} \dfrac{u}{|u|} & \text{if } |u| \neq 0, \\ |\tau_{XZ|_b}| \leq \mu_s p_T & \text{if } |u| = 0. \end{cases}$$

In this talk we present the derivation of this model, as well as a simple first order numerical scheme based on a three-step splitting procedure to solve the friction and non-hydrostatic contributions. This scheme is well-balanced for steady solutions verifying

$$u = w = q = 0, \qquad \text{and} \qquad \left|\partial_X \left(b + \widetilde{b} + h\right)\right| \leq \mu_s,$$

i.e., those solutions at rest where the slope of the free surface is lower than the angle of repose of the granular material.

Some numerical tests will be presented, including a study of the influence of the coordinate system (tilted or Cartesian) and comparisons with laboratory experiments of granular collapse. Among others, an important result is the fact that the non-hydrostatic pressure allows us to properly reproduce the parabolic shape (acceleration-deceleration phases) of the velocity of the front observed in granular collapse experiments, in contrast to hydrostatic models where only a deceleration phase is observed. Finally, remark that the details of this talk can be seen in [2].

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