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Well-posedness of the free boundary problem in ideal compressible MHD with surface tension

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In this talk we present recent results [5, 6, 7, 8] obtained jointly with Tao Wang (Wuhan University).

It should be noted that the effect of surface tension is especially important for modelling magnetohydrodynamic (MHD) flows in liquid metals (see, e.g., [2] and references therein). Even for MHD modelling of large scales phenomena like those in astrophysical plasmas, where the effect of surface tension and diffusion is usually neglected, it is still useful to keep surface tension as a stabilizing mechanism in numerical simulations of magnetic Rayleigh–Taylor instability [4].

We first consider the free-boundary ideal compressible magnetohydrodynamic (MHD) equations with surface tension governing the dynamics of inviscid, compressible, and electrically conducting fluids in three spatial dimensions. Let $\Omega(t) := \{x \in \mathbb{R}^3 : x_1 > \varphi(t, x')\}$ be the changing volume occupied by the conducting fluid at time t, where $x' := (x_2, x_3)$ is the tangential coordinate. The free boundary problem reads.

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \quad \text{in } \Omega(t), \tag{1a}$$

$$\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v - H \otimes H) + \nabla q = 0 \quad \text{in } \Omega(t), \tag{1b}$$

$$\partial_t H - \nabla \times (v \times H) = 0 \quad \text{in } \Omega(t), \tag{1c}$$

$$\partial_t (\rho E + \tfrac{1}{2}|H|^2) + \nabla \cdot (v(\rho E + p) + H \times (v \times H)) = 0 \quad \text{in } \Omega(t), \tag{1d}$$

$$\partial_t \varphi = v \cdot N \quad \text{on } \Sigma(t), \tag{1e}$$

$$q = \mathfrak{s}\mathcal{H}(\varphi) \quad \text{on } \Sigma(t), \tag{1f}$$

$$(\rho, v, H, S, \varphi)|_{t=0} = (\rho_0, v_0, H_0, S_0, \varphi_0), \tag{1g}$$

supplemented with the constraints

$$\nabla \cdot H = 0 \quad \text{in } \Omega(t),$$
$$H \cdot N = 0 \quad \text{on } \Sigma(t)$$

on the initial data, for $\partial_t := \frac{\partial}{\partial t}$ and $\nabla := (\partial_1, \partial_2, \partial_3)^\mathsf{T}$ with $\partial_i := \frac{\partial}{\partial x_i}$, where the density ρ, velocity $v \in \mathbb{R}^3$, magnetic field $H \in \mathbb{R}^3$, specific entropy S, and interface function φ are to be determined. Symbols $q = p + \frac{1}{2}|H|^2$ and $E = e + \frac{1}{2}|v|^2$ stand for the total pressure and specific total energy, respectively, where p is the pressure and e is the specific internal energy. The thermodynamic variables ρ and e are given smooth functions of p and S satisfying the Gibbs relation

$$\vartheta \, \mathrm{d}S = \mathrm{d}e + p \, \mathrm{d}\left(\frac{1}{\rho}\right),$$

where $\vartheta > 0$ is the absolute temperature. We denote by $\Sigma(t) := \{x \in \mathbb{R}^3 : x_1 = \varphi(t, x')\}$ the moving vacuum boundary, by $N := (1, -\partial_2\varphi, -\partial_3\varphi)^\mathsf{T}$ the normal vector to $\Sigma(t)$, by $\mathfrak{s} > 0$ the constant coefficient of surface tension, and by $\mathcal{H}(\varphi)$ twice the mean curvature of the boundary, that is,

$$\mathcal{H}(\varphi) := \mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'}\varphi}{\sqrt{1 + |\mathrm{D}_{x'}\varphi|^2}} \right) \quad \text{with } \mathrm{D}_{x'} := \begin{pmatrix} \partial_2 \\ \partial_3 \end{pmatrix}.$$

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The boundary condition (1e) states that the free boundary moves with the velocity of the conducting fluid and makes the free surface $\Sigma(t)$ *characteristic*. The boundary condition (1f) results from surface tension and zero vacuum magnetic field.

We establish the local-in-time existence and uniqueness of smooth solutions to the free-boundary problem (1) by a suitable modification [6] of the Nash–Moser iteration scheme. The main ingredients in proving the convergence of the scheme are the tame estimates and unique solvability of the linearized problem in the anisotropic Sobolev spaces H^m_* for m large enough. In order to derive the tame estimates, we make full use of the boundary regularity enhanced from the surface tension. The unique solution of the linearized problem is constructed by designing some suitable ε–regularization and passing to the limit $\varepsilon \to 0$. Unlike the case of zero surface tension studied in our work [5], we do not assume the fulfilment of the generalized Rayleigh–Taylor sign condition on the total pressure. That is, we rigorously prove that the surface tension has a stabilization effect on the evolution as for the cases without magnetic fields (see, e.g., [1]).

We also consider a more complicated statement [8] of the free boundary problem in MHD with surface tension. Namely, the vacuum magnetic field $\mathcal{H}$ is now not zero and satisfies the div-curl system of pre-Maxwell dynamics:

$$\nabla \times \mathcal{H} = 0, \quad \nabla \cdot \mathcal{H} = 0 \qquad \text{in } \Omega^-(t), \tag{2}$$

where $\Omega^-(t) := \{x \in \mathbb{R}^3 : -1 < x_1 < \varphi(t,x'),\, x' \in \mathbb{T}^2\}$ is the vacuum region whereas the domain $\Omega(t)$ now reads: $\Omega(t) := \{x \in \mathbb{R}^3 : \varphi(t,x') < x_1 < 1,\, x' \in \mathbb{T}^2\}$. Instead of the boundary condition (1f) we have the condition

$$[q] = \mathfrak{s}\mathcal{H}(\varphi) \quad \text{on } \Sigma(t),$$

where $[q] = q|_\Sigma - \frac{1}{2}|\mathcal{H}|^2_{|\Sigma}$ and the interface $\Sigma(t)$ now reads: $\Sigma(t) := \{x \in \mathbb{R}^3 : x_1 = \varphi(t,x'),\, x' \in \mathbb{T}^2\}$. Moreover, the vacuum magnetic field should satisfy the boundary condition

$$\mathcal{H} \cdot N = 0 \quad \text{on } \Sigma(t)$$

and we prescribe the boundary conditions on the fixed top and bottom boundaries $\Sigma^\pm := \{x \in \mathbb{R}^3 : x_1 = \pm 1,\, x' \in \mathbb{T}^2\}$:

$$v_1 = H_1 = 0 \quad \text{on } \Sigma^+, \quad \nu \times \mathcal{H} = \mathfrak{J} \qquad \text{on } \Sigma^-(t), \tag{3}$$

where $\nu = (-1, 0, 0)$ and $\mathfrak{J}$ is a given surface current. Note that the well-poseness of such a free boundary problem for the case of zero surface tension was proved in [3] under the so-called non-collinearity condition for the magnetic fields on the free interface. For this problem we justify that the surface tension again plays a stabilization role, and we assume neither the non-collinearity condition nor the Rayleigh–Taylor sign condition.

At last, we refer to our result in [6] where the stabilization effect of surface tension has been rigorously confirmed for the free boundary problem for MHD contact discontinuities.

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