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Traveling-wave solutions to reaction-convection equations with Perona-Malik diffusion

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The Perona-Malik equation is a nonlinear forward-backward parabolic equation introduced for noise reduction and edge detection of digitalized images. In one space dimension, this equation reduces to

$$u_t = [G(u_x)]_x\,, \tag{1}$$

for $t \geq 0$, $x \in \mathbb{R}$, so that $G'(u_x)$ plays the role of the diffusion coefficient. The function G is bounded with $G(\pm\infty) = 0$, and *non-monotone*, with $G' > 0$ in an interval $(-\kappa, \kappa)$ and $G' < 0$ elsewhere, see Figure 1. Thus, the diffusion is weak and negative when u_x is large and strong and positive when u_x is small.

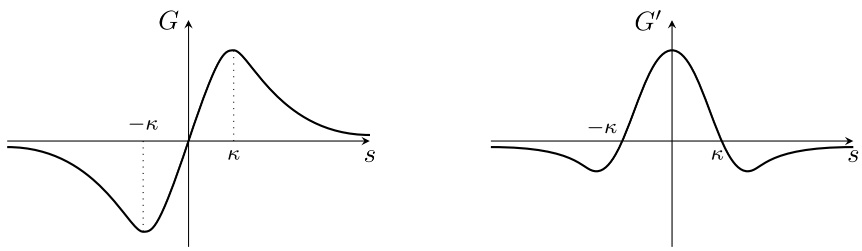

Figure 1: Plots of a typical function G (left) and its derivative G' (right).

A convection term can be added to (1) to obtain the more general equation

$$u_t + [H(u)]_x = [G(u_x)]_x\,. \tag{2}$$

In image inpainting, the convective term in equation (2) is a rough simplification motivated by more complex models [2], see also [4]. Significant efforts have been devoted to investigate traveling wave solutions for equations of the form (2), see [4, 5, 7]. We recall that a traveling wave u to (2) is a solution of the form $u(x,t) = \varphi(x - ct)$, for some $c \in \mathbb{R}$; here, we consider the case φ is smooth, global and monotone, i.e., we deal with *wavefronts*. Wavefronts for (2) are proved to exist only if the profiles φ satisfy $|\varphi'(\xi)| \leq \kappa$ for every $\xi \in \mathbb{R}$ (namely, they are *subcritical* if the inequality is strict, and *critical* otherwise). As a consequence, for these solutions the diffusion coefficient is always greater or equal to 0.

Here, we deal with the following equation, where also a reaction term is included:

$$u_t + [H(u)]_x = [G(u_x)]_x + f(u), \tag{3}$$

where $f \in C[0,1]$ satisfies $f(0) = f(1) = 0$ and $f > 0$ in $(0,1)$.

From the image processing viewpoint, the introduction of f results in an image contrast-enhancing, see [1]. In general one may assume that f has a finite number of zeros in $(0,1)$ to extract different quantized gray-levels.

Our main results [3] are the following:

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- There is a threshold speed $c_0 \in \mathbb{R}$ such that, if $c \geq c_0$, then equation (3) has a unique (up to horizontal shifts) wavefront $u \in C^1(\mathbb{R}^2)$ with wave speed c and monotone profile φ satisfying $\varphi(-\infty) = 1$ and $\varphi(+\infty) = 0$. These profiles φ are either subcritical or critical.
- The same result also holds in the case case $c < c_0$ if for the speed c_0 there exists a subcritical profile.
- We study the strict monotonicity of the profiles with respect to the ξ variable as well as with respect to the parameter c, their smoothness, and the lack of sharp (non-smooth) behavior at the equilibria.

The proof mixes comparison techniques with other tools of nonlinear analysis, such as, for instance, Schauder fixed-point theorem. Analogous results hold in the case $\varphi(-\infty) = 0, \varphi(+\infty) = 1$.

By formal computations, supercritical wavefronts for (2) are expected to exist; this guess has been verified with numerical simulations, but wavefronts are no more continuous in this case, see [4, 6].

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