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An entropy conservative and exact discontinuity capturing discrete kinetic scheme for scalar conservation laws

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The hyperbolic conservation laws with entropy pairs (entropy and entropy flux functions) satisfy entropy conservation laws in smooth regions and entropy inequalities in discontinuous regions. Discretisation of these hyperbolic conservation laws over a domain need not necessarily lead to discretisation of their corresponding entropy conservation laws and entropy inequalities. The necessary condition on interface fluxes in finite volume method (FVM) for satisfaction of discrete entropy conservation was introduced by Tadmor in [1], and the class of schemes that satisfy this condition came to be known as entropy conservative schemes.

Shock capturing schemes which can capture grid-aligned steady discontinuities exactly have been quite successful in CFD, *e.g.,* [2, 3, 4] apart from a few more. Typically, this is based on enforcing the satisfaction of Rankine-Hugoniot (R-H) jump conditions directly or indirectly in the discretisation process.

In this work, a discrete kinetic entropy conservative scheme is introduced together with exact shock capturing based on R-H conditions. The kinetic entropy pairs utilised are based on the formulation as in [5]. To switch between discrete kinetic entropy conservation (in smooth regions) and exact discontinuity capturing (at discontinuities), discrete kinetic entropy distance of [6] has been used, which incidentally also coincides with the macroscopic definition of [7].

Numerical results for a standard test problem from [8] obtained using the formulated scheme is shown in fig. 1. The problem is governed by one dimensional inviscid Burgers' equation, position variable $x \in [0, 1]$ and initial condition is $u(x, 0) = sin(2\pi x)$. The domain [0,1] is split up into 80 identical cells for computing the numerical solution. Exact solution is found using the method of characteristics. Figure 1 shows the sequential behaviour of u over time t. The initial sinusoidal wave profile at $t = 0$ is shown in fig. 1a. A smooth compressed profile formed at $t = \frac{0.8}{2\pi}$ is shown in fig. 1b. Over time, the profile compresses and forms a discontinuity at $t = \frac{1}{2\pi}$ as shown in fig. 1c. The small discontinuity beginning to form at $x = 0.5, t = \frac{1}{2\pi}$ gradually sharpens and becomes a full-fledged jump between $u = 1$ and $u = -1$ at $t = 0.25$ as shown in fig. 1d. At this time, the propagation speed of shock vanishes and stationary shock is formed. At later times, the magnitude of the discontinuity decreases and becomes a jump between $u < 1$ and $u > -1$ as shown in figs. 1e and 1f.

Two dimensional test problems from [9] governed by $\partial_t u + \partial_{x_1} g_1(u) + \partial_{x_2} g_2(u) = 0$ with $g_1(u) = \frac{1}{2}u^2$ and $g_2(u) = u$ are shown in figs. 2 and 3. For both the problems, the domain is $[0, 1] \times [0, 1]$, and the computational domain is obtained by splitting this into 65×65 identical cells. The steady state boundary conditions for normal shock problem in fig. 2 are $u(0, x_2) = 1$ for $0 < x_2 < 1$, $u(1, x_2) = -1$ for $0 < x_2 < 1$ and $u(x_1, 0) = 1 - 2x_1$ for $0 < x_1 < 1$; and the steady state boundary conditions for oblique shock problem in fig. 3 are $u(0, x_2) = 1.5$ for $0 < x_2 < 1$, $u(1, x_2) = -0.5$ for $0 < x_2 < 1$ and $u(x_1, 0) = 1.5 - 2x_1$ for $0 < x_1 < 1$.

It can be seen from these results that the formulated scheme comprising of discrete kinetic entropy conservative and exact shock capturing solvers is efficient over both smooth and discontinuous regions.

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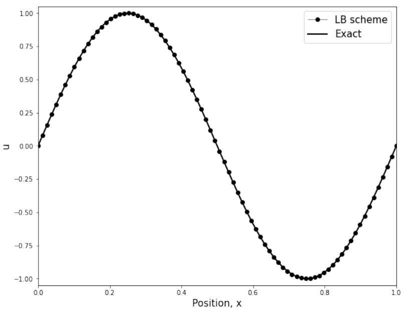
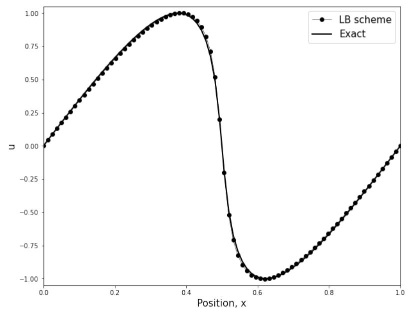
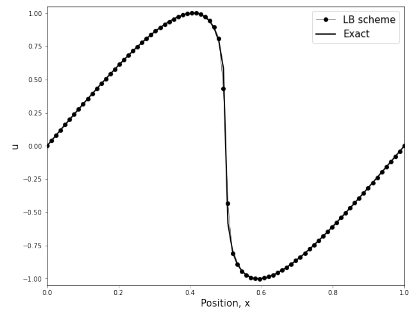


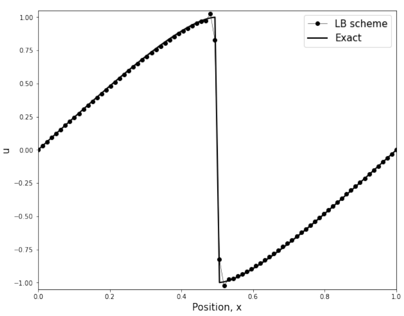
(a) Sine wave profile at $t = 0$



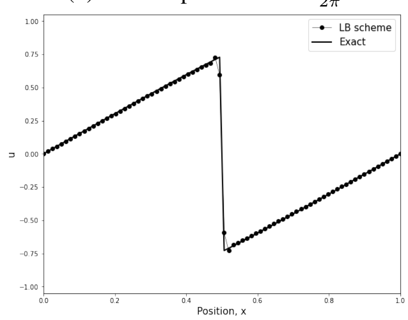
(b) Smooth profile at $t = \frac{0.8}{2\pi}$



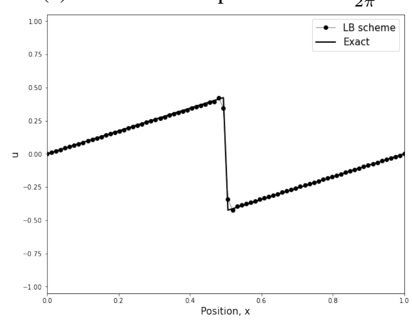
(c) Discontinuous profile at $t = \frac{1}{2\pi}$



(d) Discontinuous profile at $t = 0.25$



(e) Standing shock at $t = 0.5$



(f) Standing shock at $t = 1$

Figure 1: Time evolution of initial Sine wave profile governed by inviscid Burgers' equation

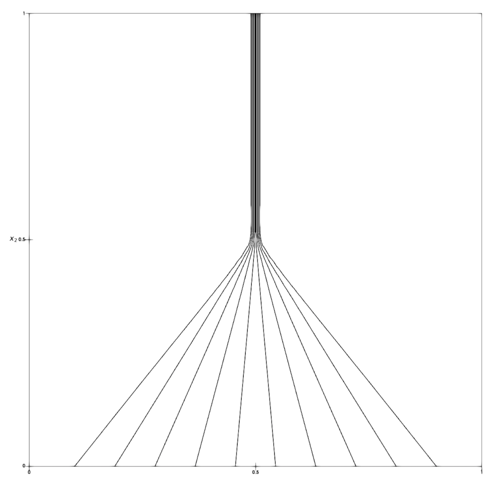


Figure 2: Normal shock

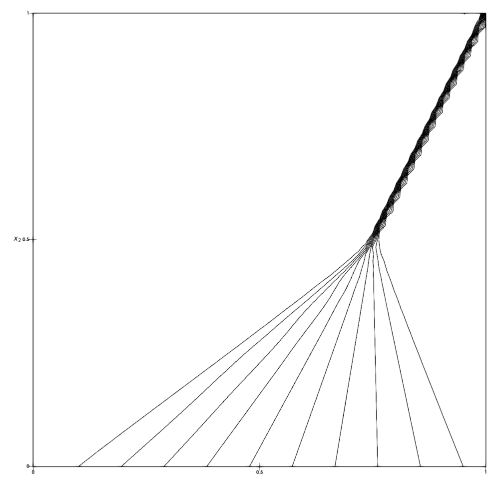


Figure 3: Oblique shock

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