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Numerical Simulations of Shallow Water Equations with Uncertainty

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We consider the Saint-Venant system of shallow water equations, which is widely used for modeling flows in rivers, lakes and coastal areas, as well as in models emerging in oceanography and atmospheric sciences. We study the Saint-Venant system with uncertainties, which may appear in the source terms, initial or boundary data due to empirical approximations or measuring errors. Quantifying these uncertainties is important for many applications since it helps to conduct sensitivity analysis and provide guidance for improving the models.

In the one-dimensional (1-D) case, the studied system reads as

$$\begin{cases} h_t + q_x = 0, \\ q_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x = -ghZ_x, \end{cases} \tag{1}$$

where the water depth $h(x,t,\xi)$, mean velocity $u(x,t,\xi)$ and water discharge $q := hu$ are functions of the spatial variable x, time t and random variable $\xi \in \Xi \subset \mathbb{R}$. In the current setting, the bottom topography $Z(x,\xi)$ is independent of time and g is the acceleration due to gravity.

It is well-known that the shallow water system (1) is hyperbolic since the Jacobian of its flux has two distinct real eigenvalues $\lambda_\pm = u \pm \sqrt{gh}$ as long as $h > 0$. It is also easy to show that (1) admits a family of smooth steady-state solutions,

$$q(x,\xi) = C_1(\xi), \quad \frac{u(x,\xi)^2}{2} + g\left(h(x,\xi) + Z(x,\xi)\right) = C_2(\xi), \tag{2}$$

Among all of the solutions in (2), the simplest "lake at rest" equilibria satisfying

$$q(x,\xi) \equiv 0, \quad w(x,\xi) := h(x,\xi) + Z(x,\xi) = C(\xi) \tag{3}$$

are often physically relevant since in many practical situations the water waves can be viewed as small perturbations of a "lake at rest" state. Capturing such waves numerically is a challenging task because the magnitude of these perturbation may be smaller that the size of the truncation error, especially when a practically affordable course mesh is used. In order to overcome this difficulty one needs to develop a so-called well-balanced scheme, which is capable of exactly preserving the steady-states solutions (2) (or at least the "lake at rest" states (3)) at the discrete level.

Another difficulty associated with the numerical solution of the studied system (1) is the dependence of its solutions on the random variable ξ representing the uncertainty present in the model. In recent years, a wide variety of uncertainty quantification methods for nonlinear hyperbolic systems has been proposed and investigated. One of the popular class methods employ Monte Carlo-type simulations (see, e.g., [5, 4, 1, 3, 8] and references therein), which are robust, but not very efficient due to a large number of realizations required. In addition to the Monte Carlo methods, a widely used approach for solving PDEs with uncertainties is the generalized polynomial chaos (gPC), where stochastic processes are represented in terms of orthogonal polynomials series

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of random variables; see, e.g., [8, 6, 1] and references therein. There are two types of gPC methods: intrusive and non-intrusive ones. In non-intrusive algorithms, like stochastic collocation methods one seeks to satisfy the governing equations at a discrete set of nodes in the random space and then use interpolation and quadrature rules to numerically evaluate statistical moments. Therefore, in the stochastic collocation approach, as well as in the Monte-Carlo methods, one can use a well-balanced numerical method designed for the corresponding deterministic Saint-Veinant system. In the case of an intrusive approach, like stochastic Galerkin (SG) methods, gPC expansions are substituted into the governing equations and projected by a Galerkin approximation to obtain deterministic equations for the expansion coefficients; see, e.g., [2, 7, 8, 6] and references therein. Solving the coefficient equations gives the stochastic moments of the solution of the original uncertain problem. The equations for the expansion coefficients are almost always coupled and thus one needs to develop new methods for their numerical solution, which may be a challenging task. Nevertheless, the gPC-SG methods are in general more accurate than their non-intrusive counterparts when the same number of modes in the gPC expansion is used and therefore a higher accuracy of the numerical solution can be achieved with a lower degree of the gPC-SG expansion.

In this talk, we develop a gPC-SG method for the Saint-Venant system (1). It is well-known that such approximations for nonlinear hyperbolic systems do not necessarily yield globally hyperbolic systems: their Jacobians may contain complex eigenvalues and thus trigger instabilities and ill-posedness. We introduce a robust, highly accurate and provable hyperbolicity-preserving gPC-SG method for the Saint-Venant system with uncertainties. In the new method, the Galerkin coefficient system is solved solved by a well-balanced positivity-preserving finite-volume method, which employs transformations between conservative (h, hu) and primitive (h, u) variables in the gPC expansions, efficient computation of the Jacobian eigenstructure, and specific reconstruction and adaptive techniques. All together these techniques ensure (including a theoretical proof) the positivity- and hyperbolicity-preserving property of the method. A number of numerical experiments will be presented to illustrate the robustness of the new approach.

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