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Entropy stable schemes for parabolic degenerate equations with a discontinuous flux

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Entropy stable schemes for hyperbolic conservation laws were introduced by Tadmor [1] and were extended in [2] to degenerate parabolic equations with continuous advective flux. In this work, we propose an extension of entropy stable schemes for (possibly strongly) degenerate parabolic equations with a discontinuous flux

(1) $$u_t + f(\gamma(x), u)_x = A(u)_{xx}, \quad (x,t) \in \Pi_T := \mathbb{R} \times [0,T],$$

with initial condition

(2) $$u(x,0) = u_0(x), \; x \in \mathbb{R},$$

where u is the function of interest (dependent on the time t and the space x) that represents a physical property of a system (mass,velocity, energy, etc.). The function f represents the convective flux of the studied physical property, γ is a function that models velocity variations dependent on space, and A represents the diffusive flux of the studied property. A diffusion is considered degenerate (strongly degenerate) when $A'(u)$ vanishes at one point (or over an entire interval $[\alpha, \beta]$).

Equation (1) is used to model processes such as: flow in porous media, sedimentation-consolidation process, and traffic flux where the flux depends discontinuously on the position; for instance, due to the driver's reaction time or variations of the road surface conditions [3, 4].

When there are discontinuities in the coefficients or in the initial condition, or the diffusion is degenerate, classical solutions of the Cauchy problem (1)-(2) do not exist. For such reason, it is necessary to seek for solutions in a variational sense which are known as weak solutions. To guarantee existence of weak solutions for this Cauchy problem, similar conditions to these are assumed:

(H1) $\gamma(x) \in BV(\mathbb{R})$.

(H2) $\gamma(x)$ is a piecewise function $C^1(\mathbb{R})$ with a finite number of jump discontinuities and $\gamma(x) \neq 0$ for all $x \in \mathbb{R}$.

(H3) $f(\gamma(x), u)$ is Lipschitz continuous in each variable and $f(\gamma(x), 0) \in L^2(\mathbb{R} \times \mathbb{R})$.

(H4) $A(u)$ is Lipschitz continuous, piecewise smooth and satisfies $A(u) \leqslant A(w)$ if $u \leqslant w$.

(H5) Intervals $[\alpha, \beta]$ where A vanishes are allowed.

(H6) $\underline{u} \leqslant u_0(x) \leqslant \overline{u}$, $\forall x \in \mathbb{R}$ for certain values $\underline{u}$ and $\overline{u}$.

Since weak solutions are not unique, extra conditions must be included to obtain uniqueness of solution for the problem of interest. This is the case of the entropy condition which is derived based on the changes of physical properties of the system when crossing a shock wave. Thus, in most cases it is not possible to obtain the analytical solution, so a numerical solution is required. These approximations are based mostly on upwind methods, relaxation schemes, and front tracking schemes [5]. All of these methods are of first order of convergence. In particular, upwind schemes have been used to solve problem (1)-(2) and its convergence to the Krûzkov entropy solution has been proved analytically [4, 6].

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In this work, we present the construction, convergence and efficiency of an entropy stable scheme for the degenerate parabolic problem (1)-(2). Entropy stable methods are second-order accurate approximations, which intrinsically guarantee a discrete entropy condition. Using the compensated compactness methodology the convergence of the proposed entropy stable scheme to a general entropy weak solution is obtained. From this study, we obtain a relationship between the artificial viscosity parameter of the entropy stable scheme and the mesh parameters to guarantee convergence. In order to test the efficiency of our method, we solve some particular examples of problem (1)-(2) using the entropy stable and upwind approximations and also analyze numerically the order of convergence. From the numerical simulations, we can conclude that the entropy stable scheme captures very well different dynamics like shock waves, rarefaction waves and oscillating solutions, improving in some cases the upwind approximation.

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