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Quinpi, or constructing implicit high order schemes for hyperbolic systems

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This talk is concerned with the challenges of writing high order implicit schemes for hyperbolic systems. The name of the game is: be linear as much as possible, while avoiding spurious oscillations, even on large time steps. In this talk, we propose a possible approach, named Quinpi.

When integrating a system of hyperbolic equations with a numerical scheme, one chooses the time step as $\Delta t = \min\left(\Delta t_{\text{acc}}, \Delta t_{\text{stab}}\right)$, where Δt_{acc} is fixed by accuracy constraints, and Δt_{stab} is due to the CFL stability condition, with

$$\Delta t_{\text{stab}} \leq c \frac{h}{||f'(u)||_S},$$

where $||f'(u)||_S$ is the spectral norm of the Jacobian of the flux. When $||f'(u)||_S \gg 1$, the stability constraint is more demanding than accuracy, and implicit schemes may become convenient.

Our ideas stem from [1] where a third order non oscillatory implicit scheme for conservation laws is presented, based on space-time limiting, and [2] where the main idea is to use a predictor to avoid some of the non linearities which appear in [1]. However, the predictor used in that work is explicit, and thus cannot provide on time information for large time steps. In our work [3], we propose a class of schemes which are based on the following steps,

- Build implicit schemes for stiff hyperbolic systems with unconditional stability, based on diagonally implicit Runge Kutta schemes (DIRK).
- Prevent spurious oscillations in space using *exactly the same* techniques for *space limiting* we are familiar with. In other words, following the method of lines, we will rewrite the PDE as a system of ODE's in the cell averages, with a non oscillatory right hand side.
- The non-linearities in the space reconstruction will be tackled using first order predictors, that can be computed without limiting, because they are unconditionally TVD. They allow to compute each DIRK stage by solving a system which is nonlinear only because of the nonlinear flux function, and, at the same time, they will constitute a low order non oscillatory approximation of the solution in its own right.
- Since a high order Runge Kutta scheme can be viewed as a polynomial reconstruction in time, not surprisingly, limiting is needed also in time, as pointed out by several authors, see for instance [1]. Unlike these authors, however, we limit the solution in time applying the limiter directly on the computed solution, blending the accurate solution with the low order predictor, without coupling space and time limiting.

On scalar conservation laws, Δt_{acc} and Δt_{stab} have the same size, because there is only one propagation speed. Implicit integrators become of interest when there are fast waves one is not interested in, which however determine Δt_{stab}, while the phenomena one would like to resolve accurately are linked to slow waves, which then determine Δt_{acc}. In this case one is willing to tolerate a deterioration of the error on the fast waves, preserving accuracy on the slow waves. This is the case for instance for many solvers for low Mach or low Froude flows.

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Table 1: CPU times for each step of the SSP-RK3 explicit scheme and of the Quinpi3 implicit scheme on linear and nonlinear problems.

Cells N	SSP-RK3	Quinpi3	Ratio r_N
200	0.0021	0.0062	2.95
400	0.0033	0.0093	2.82
800	0.0061	0.0155	2.54
1600	0.0095	0.0293	3.08

(a) Linear problem.

Cells N	SSP-RK3	Quinpi3	Ratio r_N
200	0.0023	0.0103	4.48
400	0.0032	0.0148	4.62
800	0.0066	0.0229	3.47
1600	0.0091	0.0361	3.97

(b) Nonlinear problem before shock formation.

Cells N	SSP-RK3	Quinpi3	Ratio r_N
200	0.0023	0.0107	4.65
400	0.0033	0.0158	4.79
800	0.0068	0.0255	3.75
1600	0.0095	0.0566	5.96

(c) Nonlinear problem after shock formation.

In a scalar problem this improvement is not apparent. However, we can estimate the computational cost for a single time step of the explicit third order SSP Runge-Kutta scheme and of the Quinpi3 scheme. The ratio between these two costs will tell as at what CFL the implicit scheme becomes competitive.

In Table 1 we report the results obtained on the linear equation on $[-1, 1]$ and on Burgers' equation on $[0, 2]$ with a smooth initial condition. The results are obtained on a quadcore Intel Core i7-6600U with clock speed 2.60GHz.

As expected, a step of the implicit scheme is more expensive than a step of the explicit one. But several things can be noted. First of all, we expect that the implicit scheme should be faster on linear problems. In this case in fact no Newton iteration is needed, thanks to the linearity of the predictor. This is confirmed by our data. Moreover, both the explicit and the implicit schemes should have a computational cost scaling as N, and this is true for Quinpi3.

The complexity of the implicit scheme increases on a nonlinear problem. In fact, the Quinpi3 scheme requires approximately six Newton's iterations each time step. However, the number of iterations remains bounded, and, most important, it does not depend on N. The computational cost increases slightly for the non smooth problem: in this case in fact the initial guess of the first Newton's method is given by the solution obtained at the previous time step, which may not be accurate in the presence of shocks.

In any case, it is clear that the Quinpi3 scheme becomes faster for CFL numbers larger than 5 or 6 in the nonlinear case, and even before in the linear case. Since stiff problems, such as low Mach, can have CFLs of order hundreads and even more, it is to be expected that the implicit scheme can be convenient in many applications.

In this talk, we will extend to systems of conservation laws the construction of Quinpi3 schemes.

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